Information Statistics II
Lecture 2. Basic operations of mathematical morphology

Basic idea of mathematical morphology

Describing image transformations by combinations of a few basic operations

Main targets: shape-oriented operations

translation-invariant operations

Minkowski set operations

A, B sets (of vectors)

a, b vectors (which indicate the interior of image obejcts)

$A_b \equiv \{a+b \mid a \in A\}$ **translation** of A by b

$A^c \equiv \{\tilde{a} \mid \tilde{a} \notin A\}$ **complement** of A

$A \oplus B \equiv \{a+b \mid a \in A, b \in B\} = \bigcup_{b \in B} A_b$ **Minkowski set addition** of A and B

$A \ominus B \equiv \{x \mid x - b \in A, b \in B\} = \bigcap_{b \in B} A_b$ **Minkowski set subtraction** of B from A

Proof for the subtraction:

$\Rightarrow)$ $x \in A \ominus B \Rightarrow x - b \in A$ for all $b \in B$, and $x - b \in A \Rightarrow x \in A_b$.

$\therefore x \in A \ominus B \Rightarrow x - b \in A$ for all $b \in B \Rightarrow x \in A_b$ for all $b \in B \Rightarrow x \in \bigcap_{b \in B} A_b$.

$\Leftarrow)$ $x \in \bigcap_{b \in B} A_b \Rightarrow x \in A_b$ for all $b \in B$, and $x \in A_b \Rightarrow x - b \in A$.

$\therefore x \in \bigcap_{b \in B} A_b \Rightarrow x \in A_b$ for all $b \in B \Rightarrow x - b \in A$ for all $b \in B \Rightarrow x \in A \ominus B$.

There exists a relationship $A \ominus B = (A^c \oplus B)^c$, i.e. $A \oplus B$ and $A \ominus B$ are **dual** (proof in Lecture 4.)

Reflection and Minkowski set operations
Erosion and dilation

$B^s \equiv \{-b \mid b \in B\}$ **reflection** of B

$A \ominus B = \{x \mid (B^s)_x \subseteq A\}$ Minkowski set subtraction

Proof:

$B^s = \{-b \mid b \in B\} \Rightarrow (B^s)_x = \{-b + x \mid b \in B\} = \{x - b \mid b \in B\}$

From the definition of Minkowski set subtraction, $A \ominus B \equiv \{x \mid x - b \in A, b \in B\}$.

$\therefore A \ominus B = \{x \mid (B^s)_x \subseteq A\}$.

$$A \ominus B^S = \bigcap_{b \in B} A_{-b} = \{x \mid B_x \subseteq A\}$$ **erosion** of *A* by *B*

$$A \oplus B^S = \bigcup_{b \in B} A_{-b}$$ **dilation** of *A* by *B*

Opening and closing

$$A_B \equiv \left(A \ominus B^S\right) \oplus B$$ **Opening** of *A* by *B*

$$A^B \equiv \left(A \oplus B^S\right) \ominus B$$ **Closing** of *A* by *B*

Opening and closing are also dual: i.e. $A^B \equiv \left[\left(A^c\right)_B\right]^c$.

In the above discussion, both the sets *A* and *B* are equivalent. However, in many cases, *A* indicates a target image and *B* indicates a small figure that acts the role of filter window. In this case *B* is called a **structuring element.**

Table 1. Comparison of Minkowski subtraction / addition and erosion / dilation.

Minkowski subtraction $A \ominus B$	*erosion* $A \ominus B^S$
A moving along *B*	*A* moving along B^S
B^S moving along *A*	*B* moving along *A*

Minkowski addition $A \oplus B$	*dilation* $A \oplus B^S$
A moving along *B*	*A* moving along B^S
B moving along *A*	B^S moving along *A*

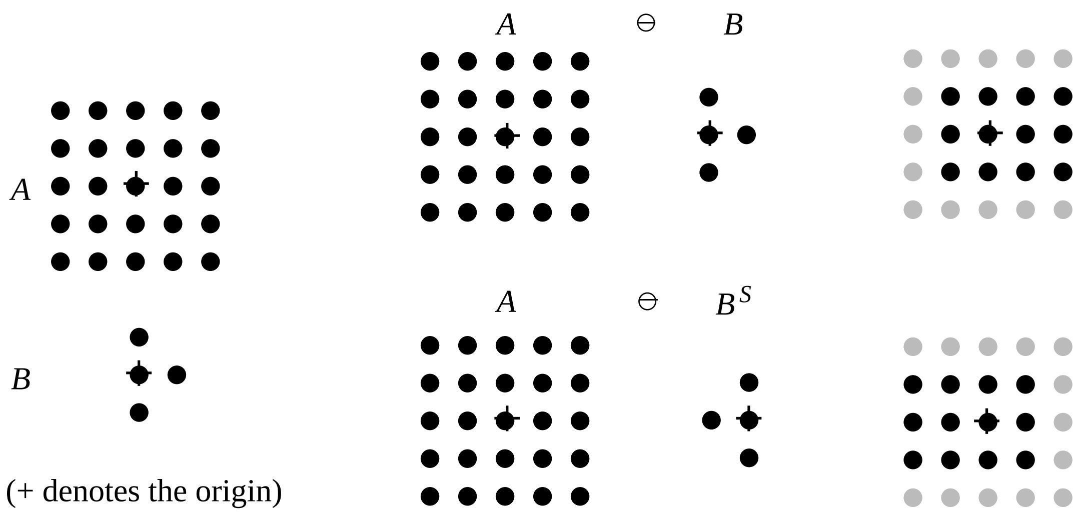

Fig. 1. Minkovski set subtraction and erosion.

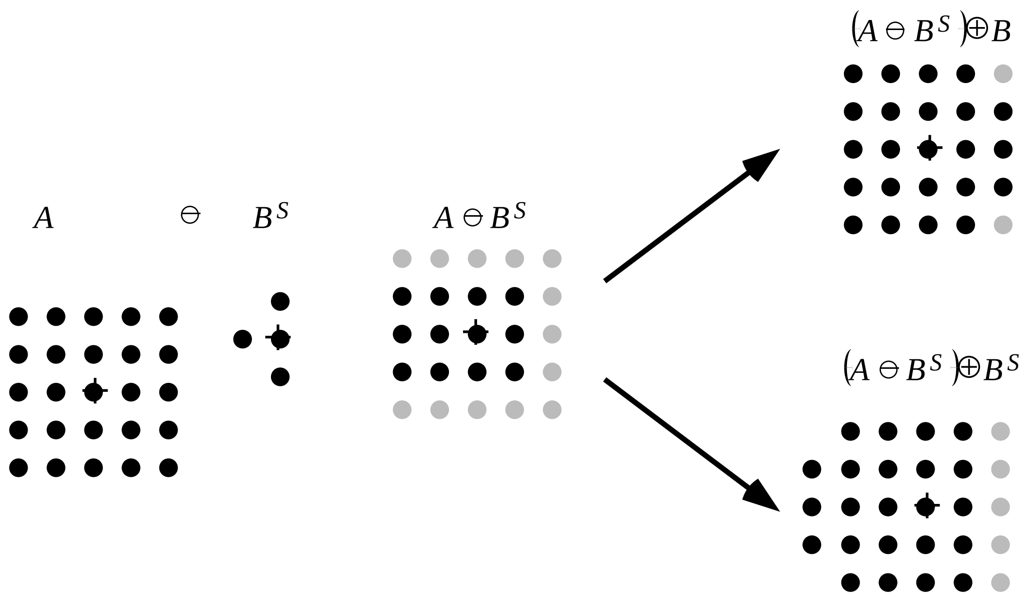

Fig. 2. Opening is not exactly "erosion followed by dilation."

Why the structuring element is reflected?

Minkovski set subtraction:	Intersection of loci of *A* moving along the vectors in *B*
Erosion:	Loci of the origin of translated *B* that included by *A*
$A_B = \cup\{B_x \mid B_x \subseteq A\}$	Description of opening (Proof in Lecture 4)

Opening removes portions smaller than the structuring element and preserves other rough shape of the original object. This property is guaranteed if they are defined using a reflected pair of the structuring elements.

"Increasing" and "idempotent"

An operation is **increasing** if the inclusion of two sets are preserved by the operation. Both opening and closing are increasing, i.e. $A \subset X \Rightarrow A_B \subset X_B$ and $A \subset X \Rightarrow A^B \subset X^B$ .This is an appropriate property for noise-removing filters.

An operation is **idempotent** if the output of the operation is not modified by further applications of the same operation. Both opening and closing are idempotent, i.e.

$\left(A_B\right)_B = A_B$ and $\left(A^B\right)^B = A^B$.

(Proofs in Lecture 4)