Information Statistics II

Lecture 3. Gray scale morphological operations

Set operations and function operations

Binary morphological operations are performed for the sets whose elements are vectors corresponding to pixel positions. These are called "set - set operations." In case of grayscale images and/or grayscale structuring elements, morphological operations are performed for "functions" from vectors to pixel vaules. Morphological operations for grayscale images and binary structuring elements are called "function-set operations," and those for grayscale images and grayscale structuring elements are called "function-function operations."

Function-set operation

$f(\boldsymbol{x})$ function to describe a grayscale image

$X_t\left[f(\boldsymbol{x})\right] \equiv \left\{x \mid f(\boldsymbol{x}) \geq t\right\}$ *cross section* of $f(\boldsymbol{x})$ with value t

NB. $t < t' \Rightarrow X_{t'}\left[f(\boldsymbol{x})\right] \subseteq X_t\left[f(\boldsymbol{x})\right]$. (*)

$X_t\left[f \ominus B^S\right] \equiv X_t[f] \ominus B^S$ *function-set erosion*

$X_t\left[f \oplus B^S\right] \equiv X_t[f] \oplus B^S$ *function-set dilation*

The eroded/dilated functions are reconstructed by the cross sections defined above, since erosion/dilation are increasing and preserve the relationship (*). This reconstruction yields the following relationships;

$\left[f \ominus B^S\right](\boldsymbol{x}) = \inf_{\boldsymbol{x} \in B_x} f(\boldsymbol{x})$ *function-set erosion*

$\left[f \oplus B^S\right](\boldsymbol{x}) = \sup_{\boldsymbol{x} \in B_x} f(\boldsymbol{x})$ *function-set dilation*

Umbra and function-function operations

Umbra is intuitively understood as the shade of a function curve. If $\boldsymbol{x}$ is an n-dimensional vector, the umbra of $f(\boldsymbol{x})$ is a set of n+1-dimensional vectors and defined as follows:

$U[f] \equiv \left\{(\boldsymbol{x}, z) \mid z \leq f(\boldsymbol{x})\right\}$ *umbra* of $f(\boldsymbol{x})$

NB . $f(\boldsymbol{x}) = \max_z U[f](\boldsymbol{x}, z)$. (**)

NB. $f(\boldsymbol{x}) \equiv -\infty$ if $\boldsymbol{x}$ is out of the support.

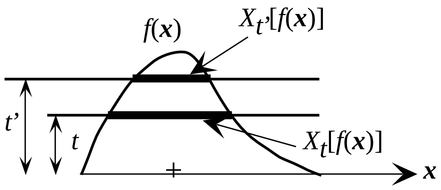

Fig. 1. Cross sections.

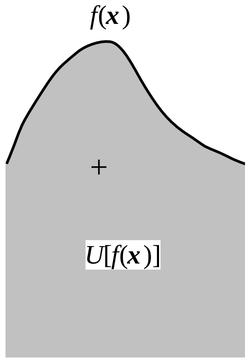

Fig. 2. Umbra.

Umbra has the following property:

$$U[f] \cup U[g] = U[max(f, g)] \tag{1}$$

$$U[f] \cap U[g] = U[min(f, g)] \tag{2}$$

Dilation of a function $f(\boldsymbol{x})$ by a function $g(\boldsymbol{x})$ is defined by dilation of $U[\ f]$ by $U[g]$, as follows:

$$U[f \oplus g^S] \equiv U[f\ ] \oplus U[g^S] = \bigcup_{y \in U[g^S]} \left[U[f\ ]\right]_y \tag{3}$$

where $g^S(\boldsymbol{x}) = g(-\boldsymbol{x})$.

Erosion of a function $f(\boldsymbol{x})$ by a function $g(\boldsymbol{x})$ is defined by dilation of the complements of umbrae $U[\ f(\boldsymbol{x})]$ and $U[g(\boldsymbol{x})]$, as follows:

$$\widehat{U}[f(\boldsymbol{x}) \ominus g^S(\boldsymbol{x})] \equiv \left[U[-f(\boldsymbol{x})] \oplus U[g^S(\boldsymbol{x})]\right]^c, \tag{4}$$

where $\widehat{U}[f(\boldsymbol{x})]$ is the inverse of an umbra and defined as $\left[U[-f(\boldsymbol{x})]\right]^c$. This definition is illustrated in Fig. 3.

The reason why the definiton by the complements should be applied is $\bigcap_{y \in U[g^S]} \left[U[f\ ]\right]_y = \phi$ since the set $U[g(\boldsymbol{x})]$ extends to $-\infty$.

Using the relationship (**), reconstruction of the function from an umbra,the definitions of dilation and erosion are interpreted into the following simple forms:

$$\left[f \oplus g^S\right](\boldsymbol{x}) = \max_{\boldsymbol{y} \in G} \left\{f(\boldsymbol{x} + \boldsymbol{y}) + g(\boldsymbol{y})\right\} \tag{5}$$

$$\left[f \ominus g^S\right](\boldsymbol{x}) = \min_{\boldsymbol{y} \in G} \left\{f(\boldsymbol{x} + \boldsymbol{y}) - g(\boldsymbol{y})\right\} \tag{6}$$

where G is the extent of g.
(*NB.* Minkovski addition is defined as follows (subtraction similarly):

$$[f \oplus g](\boldsymbol{x}) = \max_{\boldsymbol{y} \in G} \{ f(\boldsymbol{x} - \boldsymbol{y}) + g(\boldsymbol{y}) \} \tag{7}$$

Interpretation by fuzzy logic

Grayscale morphological operations are described by maximum and minimum operations as shown above. Maximum and minimum correspond to OR and AND operations, respectively, in the definition of the fuzzy logic system, if the pixel value is interpreted into the value of a membership function at each pixel. Thus binary morphological operations and grayscale operations are unified into logical opertions of pixel values.

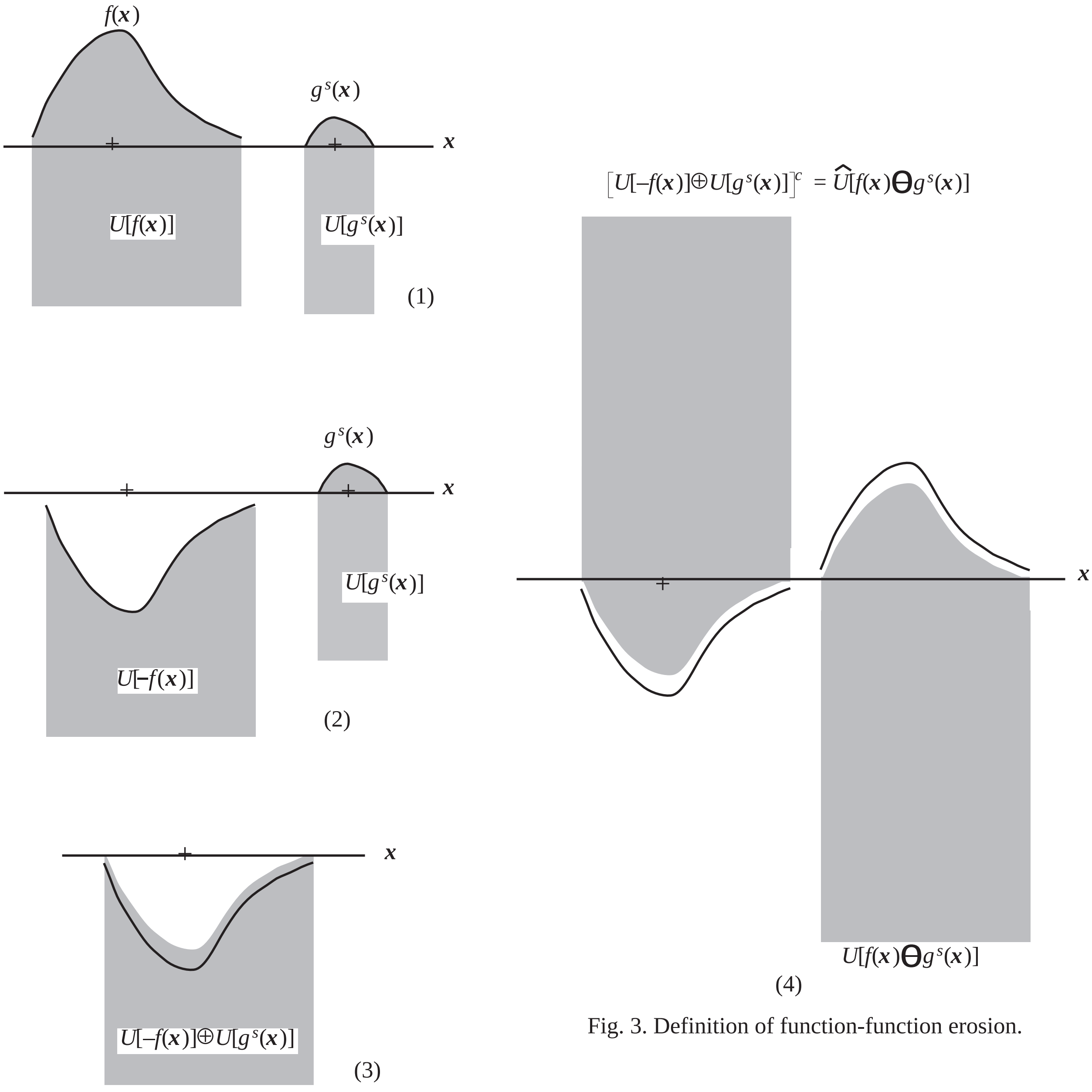

Fig. 3. Definition of function-function erosion.