Information Statistics II

Lecture 5. Granulometry, mathematical morphology in statistical sense

What is granulometry?

The *granulometry* is a series of openings of an image by structuring elements of increasing sizes. If the opening is replaced with the closing, it is called *anti-granulometry*.

What is pattern spectrum?

The pattern spectrum is the difference of the area of openings by SEs of adjacent sizes in the granulometry. The pattern spectrum of image X with structuring element B, denoted as $\mathrm{PS}_X(r, B)$ is defined as follows:

$$\mathrm{PS}_X(r, B) = \begin{cases} -dA(X_{rB}) / dr & r \geq 0 \\ dA(X^{(-r)B}) / dr & r < 0 \end{cases} \qquad (1)$$

in continouse case and in discrete case

$$\mathrm{PS}_X(r, B) = \begin{cases} A(X_{rB}) - A(X_{(r+1)B}) & r \geq 0 \\ A(X^{(-r+1)B}) - A(X^{(-r)B}) & r < 0 \end{cases} \qquad (2)$$

where $A(Y)$ denotes the area of Y.

Definition of "size"

In case of continuous images, *r times B* is defines as

$rB = \{rb \mid b \in B\}$

In case of discrete image

$rB = B \oplus B \oplus \cdots \oplus B$ (*r times*)

If B is defined as size 1, then the size of rB is r.

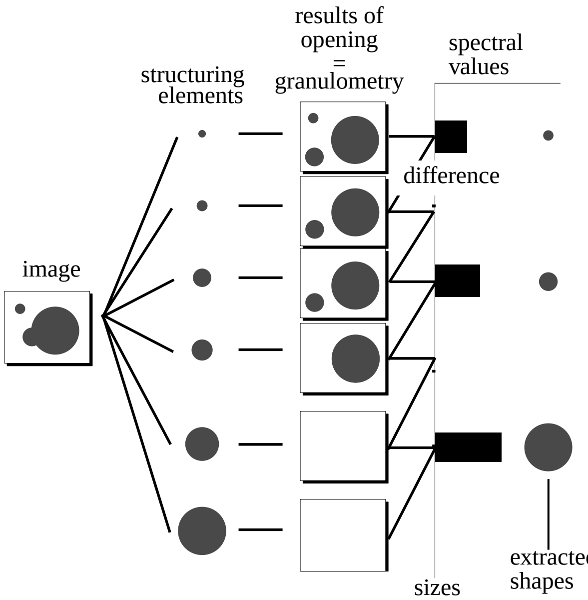


Fig.1. Granulometry and spectral values of pattern spectrum.

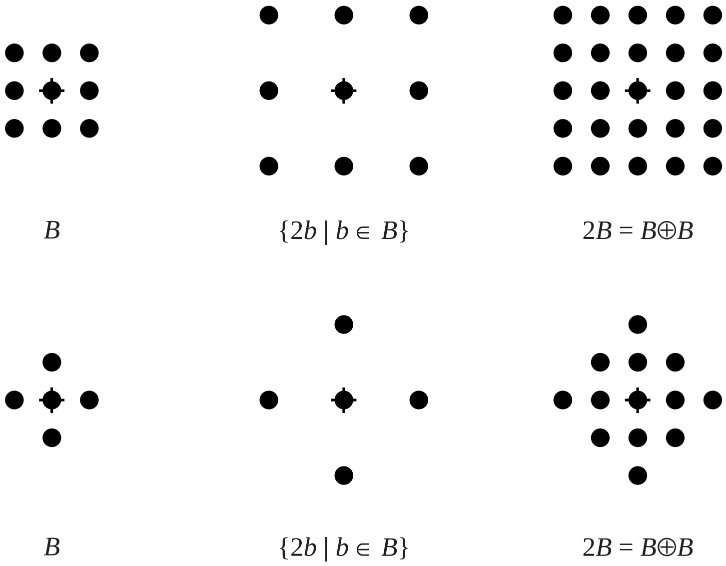
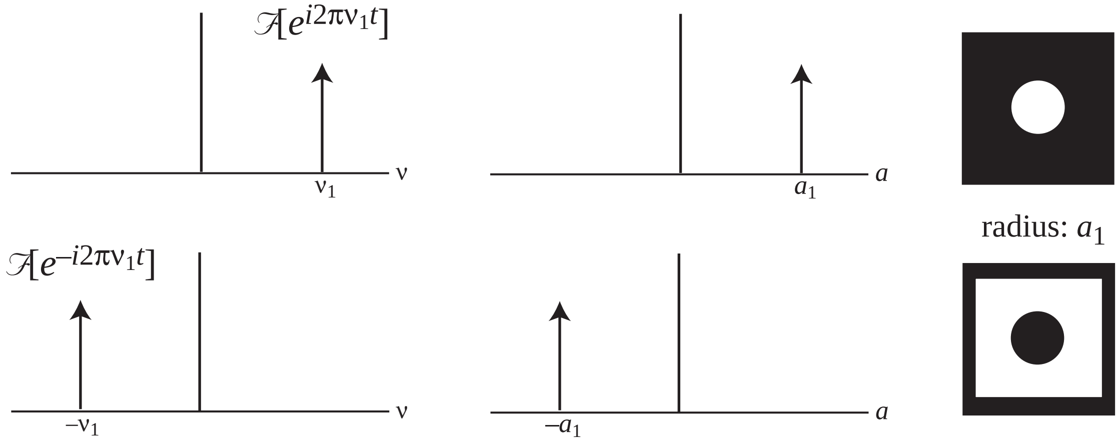


Fig.2. Definitions of size 1 and 2 in discrete case.

Comparsion of Fourier power spectrum and pattern spectrum

Fourier power spectrum	Pattern spectrum
time-series signal $s(t)$	set X descibing an object in an image
sinusoidal function $e^{-i\omega t}$ for probe	structuring element B for probe
frequency ω	size n
integration	morphological opening / closing
calculating power of complex spectral value	measuring areas of differences of granulometry

NB. It is not possible to reconstruct the original image from its pattern spectrum, as it is not possible to reconstrcut the original signal from its Fourier power spectrum.



Fourier power spectrum

Pattern spectrum

Fig.3. Comparison of Fourier power spectrum and pattern spectrum.

Interpretation in statistical sense

The *size distribution function* $F_{X,\,B}(r)$ is defined as

$$F_{X,\,B}(r) = \frac{A(X_{rB})}{A(X)} \tag{3}$$

for both continous and discrete cases. $1 - F_{X,\,B}(r)$ is regarded as a probability distribution function. The differentiation of $1 - F_{X,\,B}(r)$ is called the *size density function*, denoted as follows for $r \geq 0$:

$$p_{X,\,B}(r) = \frac{d}{dr}\left(1 - F_{X,\,B}(r)\right) = -\frac{1}{A(X)}\frac{dA(X_{rB})}{dr} = \frac{\mathrm{PS}_X(r,\,B)}{A(X)} \tag{4}$$

in continous case. In discrete case, this is regarded as probability and derived as:

$$\begin{aligned} p_{X,\,B}(r) &= \left(1 - F_{X,\,B}(r+1)\right) - \left(1 - F_{X,\,B}(r)\right) \\ &= \frac{1}{A(X)}\left(A(X_{rB}) - A(X_{(r+1)B})\right) = \frac{\mathrm{PS}_X(r,\,B)}{A(X)} \end{aligned} \tag{5}$$

i. e. we get the same result.

The following statistics are derived from these functions (for discrete case; summation is replaced with integral for continous case), where N is the maximum size in X:

Mean: $$E(X,\,B) = \sum_{r=0}^{N} r p_{X,\,B}(r) \tag{6}$$

This value means average size of X with respect to B.

Entropy: $$H(X,\,B) = -\sum_{r=0}^{N} p_{X,\,B}(r) \log p_{X,\,B}(r) \tag{7}$$

This value means average roughness of edges in X. In case $H(X,\,B) = 0$, X contains only one size of rB and its roughness is the mimimum. In case $H(X,\,B) = \log(N+1)/(N+1)$, i. e. its maximum, X contains all sizes equally and its roughness is the maximum.

Reference

P. Maragos, "Pattern Spectrum and Multiscale Shape Representation," *IEEE Trans. Patten Anal. Machine Intell.,* **11,** 7, 701-716 (1989).