Information Statistics II

Lecture 10. Morphological filters

Meaning of "filter" in morphological sense

In morphological sense, the term *filter* is restricted to all image-image transformations thatare *translation-invariant* and *increasing*. The definition of *morphological filter* or *M-filter* sometimes requires *idempotency* in addition to the above requirement of filter.

The class of *anti-extensive* morphological filters are called *τ-opening*, and the class of *extensive* morphological filters is called *τ-closing*. The ordinary opening and closing are the most basic operations of τ-opening and τ-closing, respectively.

Filter theorem

Let $\Psi(X)$ be a filter. The filter theorem states that any filter is decomposed into OR of erosions. More formally,

$$\Psi(X) = \bigcup_{B\in Ker[\Psi]} X \ominus B^S \qquad (1)$$

for any $\Psi(X)$. Here $Ker[\Psi]$ is called *kernel* of Ψ and defined as follows:

$$Ker[\Psi] = \{X \mid \mathbf{0} \in \Psi(X)\} \qquad (2)$$

Note that the followings hold:

$$Ker[\Psi_1] = Ker[\Psi_2] \Leftrightarrow \Psi_1 = \Psi_2 \qquad (3)$$

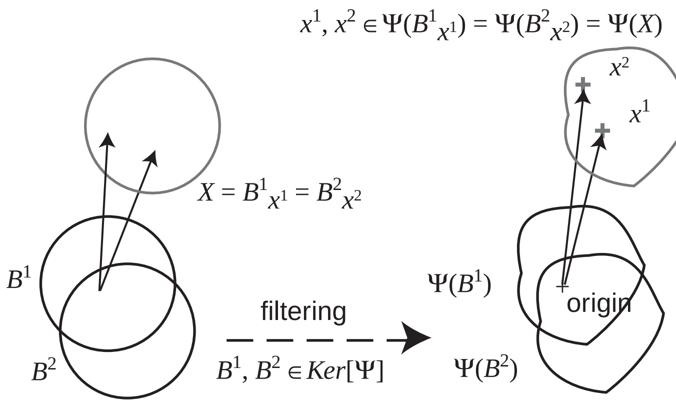
$$\Psi(X) = \{x \mid X = A_x \text{ for } \exists A \in Ker[\Psi]\} \qquad (4)$$

Equation (3) states that a filter is exactly characterized by its kernel. Equation (4) states how the filter is reconstructed from its kernel. The filter theorem, Eq.(1), is directly derived from Eq. (4).

Intuitive explanation of Eq. (4): Since A is an element of the kernel, the origin is contained by $\Psi(A)$ (= output from A). Thus x is an element of $\Psi(A_x)$ (= output from A_x), because of the translation-invariance of Ψ. Since $X=A_x$, x is an element of $\Psi(X)$ (= output from X) .

Intuitive explanation of Eq. (1):

Since $B \in Ker[\Psi]$, we get from Eq. (4) $x \in \Psi(B_x)$. If $x \in X \ominus B^S$, we get $\bigcup_{x\in X\ominus B^S} B_x = X$, and since



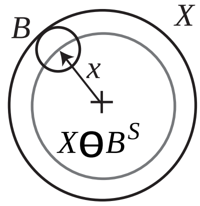


Fig. 2. Relation ship between X and B_x

Fig. 1. Intuitive explanation of Eq. (4).

Ψ is increasing, $\Psi(B_x) \subset \Psi(X)$. Thus the union of all $x \in X \ominus B^S$ for $\forall B \in Ker[\Psi]$ yields $\Psi(X)$.

Using dual of Ψ, denoted by Ψ^*, the filter also has the following expression:

$$\Psi(X) = \bigcap_{B \in Ker[\Psi^*]} X \oplus B^S \tag{5}$$

Kernels are generally redundant, and the minimum set of Ψ, called *basis* and denoted $Bas[\Psi]$, that can be replaced with the kernel can be found.

Representation theorem

Let $\Psi(X)$ be a M-filter. The family of invariant sets of Ψ, denoted $Inv[\Psi]$, is defined as:

$$Inv[\Psi] = \{X \mid \Psi(X) = X\} . \tag{6}$$

The basis of $Inv[\Psi]$, denoted β, is defined as a subset of $Inv[\Psi]$ whose elements and their translations can compose any $X \in Inv[\Psi]$.

The representation theorem denotes that any τ-opening Ψ is expressed as follows:

$$\Psi(X) = \bigcup_{B \in \beta} X_B \tag{7}$$

Intuitive explanation: X_B is the largest part of X which is composed by the union of translations of B only. Since B is an element of $Inv[\Psi]$, $\Psi(B) = B$. Since Ψ is translation-invariant and increasing, $\Psi(X_B) = X_B$. Since Ψ is anti-extensive, i. e. no points out of X appear in $\Psi(X)$, $\Psi(X)$ can be composed by $\Psi(X_B)$. Thus the union of X_B for appropriate $B \in \beta$ composes $\Psi(X)$.

Similarly, τ-closing is expressed as follows:

$$\Psi(X) = \bigcap_{B \in \beta^*} X^B \tag{8}$$

where β^* is the basis of the family of invariant sets of Ψ^*.

Examples

The median filter with window size n is decomposed into:

The maximum of the minima in all partial windows of size [n / 2 + 1]

and also

The minimum of the maxima in all partial windows of size [n / 2 + 1].

Even linear filters can be decomposed into OR of erosions. For example, the average of two pixels is expressed morphologically as

$$0.5[f(x) + f(x+1)] = \sup_{r \in R} [\min \{f(x) - r, f(x+1) + r\}] . \tag{9}$$

Practical morphological filters

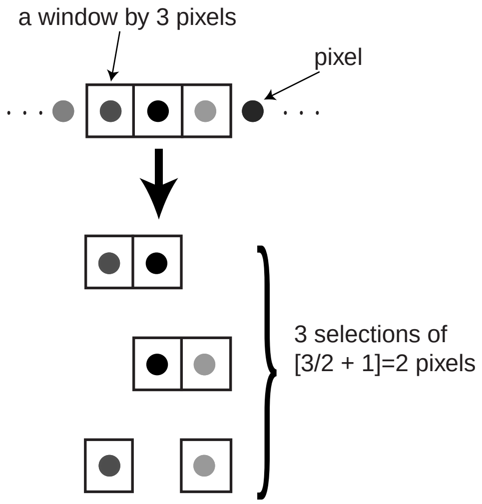


Fig. 3. Selections of pixels in a window.

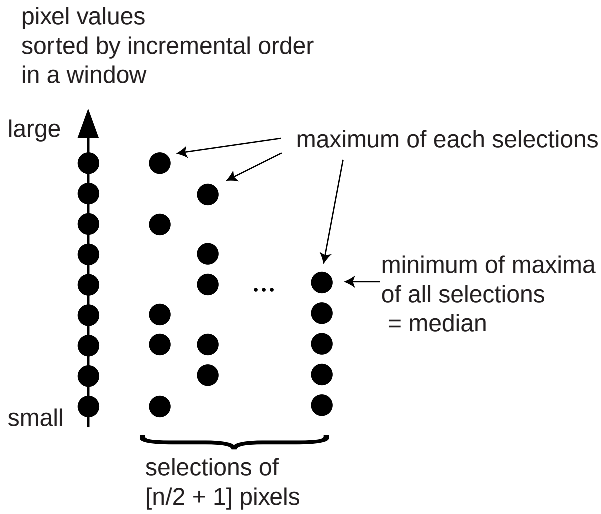


Fig. 4. Illustration of expressing median by minimum of maxima.

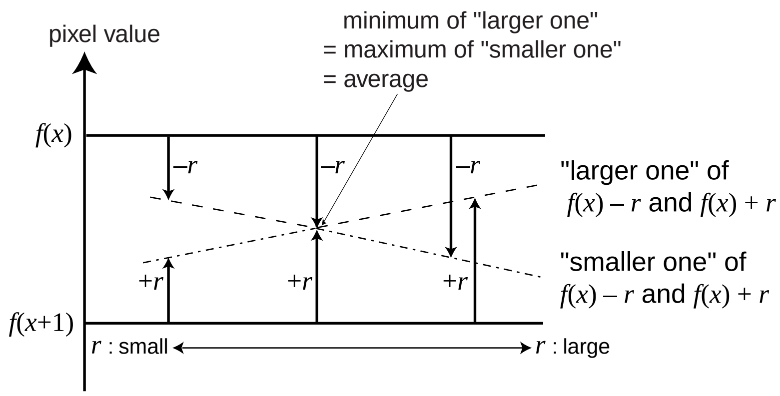


Fig. 5. Illustration of expressing average by max of mins or min of maxs.

Extensivity and anti-extensivity are not always useful fo practical situations. For example, the median filter is not extensive or anti-extensive but it handles "white" objects and "black" backgrounds equivalently. The cascades of opening and closing, *open-closing* and *clos-opening*, are used for this purpose.

References

for morphological representation of median and linear filters:

P. Maragos and R. W. Schafer, "Morphological Filters- Part I: Their Set- Theoretic Analysis and Relations to Linear Shift-Invariant Filters," *IEEE Trans. Acoust., Speech, Signal Processing,* **ASSP-35,** 8, 1153-1169 (1987).

P. Maragos and R. W. Schafer, "Morphological Filters- Part II: Their Relations to Median, Order-Statistic, and Stack Filters," *IEEE Trans. Acoust., Speech, Signal Processing,* **ASSP-35,** 8, 1170-1184 (1987).

for the filter theorem and the representation theorem:

間瀬　茂，上田修功，”モルフォロジーと画像解析(I)，” *信学誌，* **74,** 2, 168-174 (1991).

間瀬　茂，上田修功，”モルフォロジーと画像解析(II)，” *信学誌，* **74,** 3, 271-279 (1991).