2005 Autumn semester Pattern Information Processing
Topic 1. Sampling and digital processing of images

Session 2. (1) Spatial frequency and Fourier transformation

Organization of digital images is explained in this and the next sessions. Although an image is naturally a continuous distribution of brightness, it has to be converted into a discrete set of integers for computer processing. The conversion into discrete pixels is called *sampling,* and the conversion into integers is called *quantization.* The sampling period is quite important issue, and it is evaluated by the concept of *spatial frequency.* The concept of spatial frequency and Fourier transformation are explained in this session.

Diffraction of light and imaging

We start from the optical phenomenon of imaging. The imaging is defined as a collection of diverged light spread from a point of object into one point by a lens. This phenomenon can be observed from the following point of view: The light has the property of *diffraction.* The diffraction of wave is a phenomenon that the wave reaches beyond an opaque object obstracting its path. For example, even if the progress of wave on a water surface is obstructed by a board, it reaches beyond the board. Since the light is a kind of electromagnetic wave, the light has this property. The radio wave reaches beyond obstructing objects from the broadcasting station by diffraction.

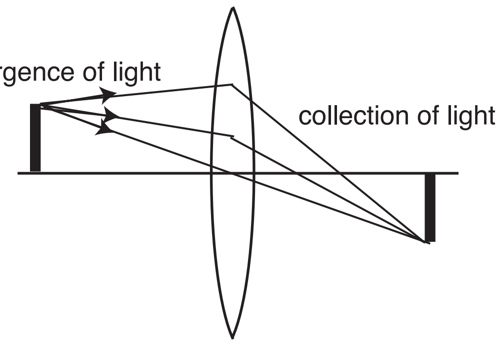

Fig. 1: Imaging.

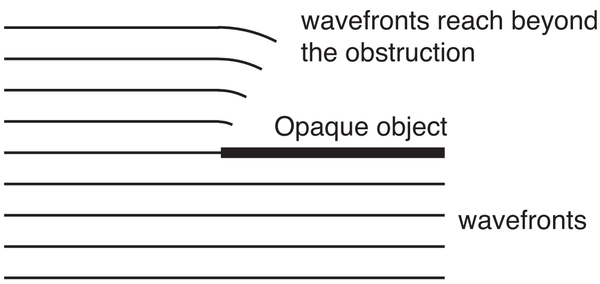

Fig. 2: Diffraction.

If the light passes through a *diffraction grating,* which is an object whose transparence is changing periodically or where transparent and opaque bands are aligned one after another, the light that passes through a transparent band interferes with the light that passes the other transparent bands. Since the light waves passing through adjacent bands emphasize each other along the direction such that the phase shift of the light waves is exactly the same as the wavelength, a bright light, called the *diffracted light of first order,* is obtained in this direction. The smaller the period of bands, the larger the angle between the diffracted light and the light passing directly through the grating (called the *zeroth order light*). If the grating contains pure opaque and pure transparent bands only, diffracted lights along several directions are obtained. However, if the transparence of grating is sinusoidally distributed, diffracted lights of zeroth and first orders only are obtained.

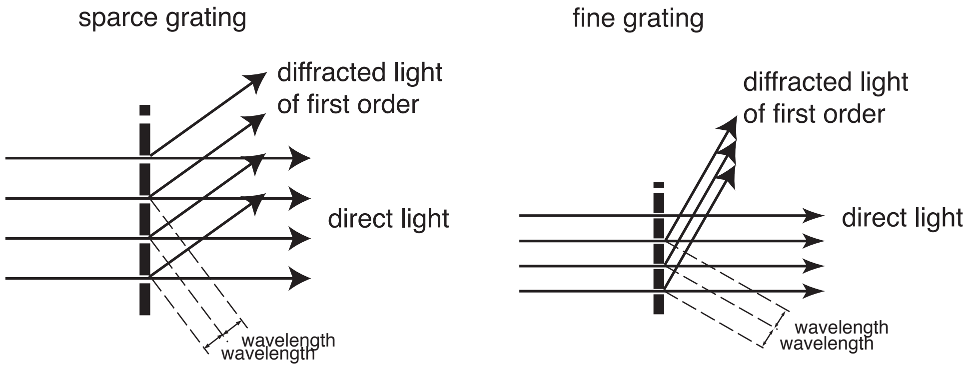

Fig. 3: Diffraction grating and diffracted light of first order.

Suppose that a figure on a transparent film is illuminated by a plain light wave[1]. Suppose also the figure is organized by a superposition of many sinusoidal wave of transparence, i.e. superposition of many diffraction gratings. Each grating diffracts the incident light and produces the diffracted light. The smaller the period of grating is, the larger the angle of diffraction is.

If these diffracted lights pass through the imaging lens, each diffracted light is refracted and interferes with the zeroth order light at the image plane. This interference produces a stripe, or *interference* fringe, on the image plane. The larger the angle of diffracted light is, the smaller the period of stripe is. The distribution of transparence on the film is reconstructed on the image plane as the superposition of these stripes.

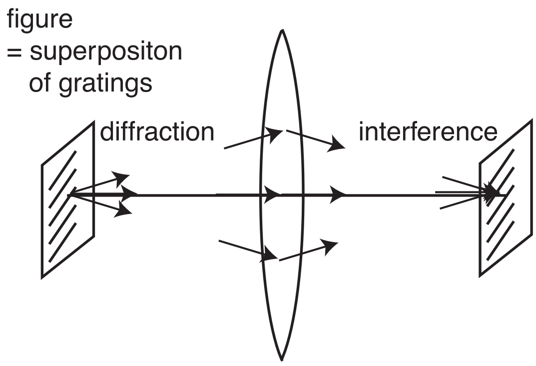

Fig. 4: Imaging by diffraction and interference.

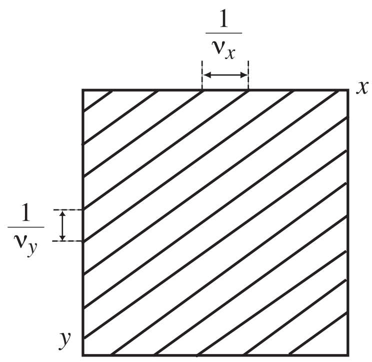

Fig. 5: Spatial frequency.

Spatial frequency

Understanding the process of imaging, the figure on the film is regarded as a superposition of sinusoidal waves of transparence. The number of repetition of the sinusoidal wave per unit length is called *spatial frequency.* The unit of spatial frequency is cycle/m in the MKSA unit system. Note that the wave is “a wave on the plane.” Since the wave on the plane

[1]The following explanation is in case of coherent illumination such as lasers. It is more complicated in the case of ordinary infoherent illumination.

has its direction, the spatial frequency is described as a set of ν_x and ν_y which are frequencies along x-direction and y-direction, respectively. The figure on a film is decomposed into a set of waves of various spatial frequencies. The amplitude of wave at a specific spatial frequency is called *component* corresponding to the spatial frequency.

Fourier transformation

It is shown in the previous section that a figure on a film can be decomposed into spacial frequency components. *Fourier transformation* is the operation to calculate the components. The principle of Fourier transformation will be explained in the following way: The transparence distribution of the figure on the film can be regarded as a mathematical function. Here we assume onedimensional functions for simplicity. The function $f(x)$ is assumed as a superposition of sinusoidal waves of various frequencies. The sinusoidal wave of frequency ν_1 is expressed in exponential form as $\exp(i2\pi\nu_1 x)$. The multiplication of 2π expresses the frequency in radian per unit length. This value, $2\pi\nu_1$, is called *angular frequency.*

This exponential function has the following property:

$$\int_{-\infty}^{\infty} \exp(i2\pi\nu_1 x)\exp(-i2\pi\nu_2 x)dx = \delta(\nu_1 - \nu_2). \quad (1)$$

The right side of Eq. (1) is called *Dirac's delta function,* defined as

$$\delta(x) = 0 (x \neq 0), \int_{-\infty}^{\infty} \delta(x) = 1. \quad (2)$$

Equation (1) states that the integral has a nonzero value only if two waves of the same frequency are superposed, otherwise it has zero. Such kind of function set is called the *orthogonal function system.* This is explained again in the section for image compression in Topic. 2.

Assuming that the function $f(x)$ is a superposition of sinusoidal waves, it can be expressed as follows:

$$\begin{aligned} f(x) = & \quad a_1 \exp(i(2\pi\nu_1 x + \theta_1)) \\ & + a_2 \exp(i(2\pi\nu_2 x + \theta_2)) + \cdots \\ & + a_n \exp(i(2\pi\nu_n x + \theta_n)) + \cdots \quad (3) \end{aligned}$$

where $\theta_1, \theta_2, \ldots \theta_n$ are phase shifts of each wave. Applying the following operation,

$$\int_{-\infty}^{\infty} f(x)\exp(-i2\pi\nu_1 x)dx, \quad (4)$$

to Eq. (3), we get

$$\begin{aligned} & \int_{-\infty}^{\infty} f(x)\exp(-i2\pi\nu_1 x)dx \\ = & \int_{-\infty}^{\infty} a_1 \exp(i(2\pi\nu_1 x + \theta_1))\exp(-i2\pi\nu_1 x)dx \\ + & \int_{-\infty}^{\infty} a_2 \exp(i(2\pi\nu_2 x + \theta_2))\exp(-i2\pi\nu_1 x)dx \\ + & \cdots + \int_{-\infty}^{\infty} a_n \exp(i(2\pi\nu_n x + \theta_n))\exp(-i2\pi\nu_1 x)dx \\ + & \cdots . \quad (5) \end{aligned}$$

From Eq. (1), we get that the first term of right side of Eq. (5) is

$$\begin{aligned} & \int_{-\infty}^{\infty} a_1 \exp(i(2\pi\nu_1 x + \theta_1))\exp(-i2\pi\nu_1 x)dx \\ = & \int_{-\infty}^{\infty} a_1 \exp(i\theta_1))\exp(i2\pi\nu_1 x)\exp(-i2\pi\nu_1 x)dx \\ = & \ a_1 \exp(i\theta_1)\delta(0) \quad (6) \end{aligned}$$

and all the other terms are zero. Consequently, Eq. (4) means the operation of extracting the amplitude of the sinusoidal wave of frequency ν_1, i. e. the component of frequency ν_1, as its real part. The imaginary part indicates the phase shift of the wave.

Applying the operation of Eq. (4) for various ν, we get various frequency components. Regarding these components as a function of ν, we denote as

$$F(\nu) = \int_{-\infty}^{\infty} f(x)\exp(-i2\pi\nu x)dx. \quad (7)$$

This is the definition of Fourier transformation, and applying this operation to the function in Eq. (3) yields peaks of heights $a_1, a_2, \ldots, a_n, \ldots$ at positions $\nu_1, \nu_2, \ldots, \nu_n, \ldots$ on the real part of ν axis, respectively. In two dimensional case, it is expressed as

$$F(\nu_x, \nu_y) = \iint_{-\infty}^{\infty} f(x, y)\exp\{-i2\pi(\nu_x x + \nu_y y)\}dxdy. \quad (8)$$

The original (x, y) plane is called real domain, and (ν_x, ν_y) plane yielded by Fourier transformation is called *frequency domain.*

Is it really able to express the original function $f(x)$ by a superposition of sinusoidal waves as Eq. (3)?

Let us assume that $f(x)$ is a periodic function of period L. In this case, the period of all the superposed waves in Eq. (3) should be also L. Thus the right side of Eq. (3) can contain the waves of basic periods $L/2, L/3, L/4, \ldots, L/n, \ldots$ only, where n is an integer. The infinite number of such waves can be composed, however, the basic periods are discrete. Thus the number of such waves is the countable infinity, and it is possible to write the function by a sum of the infinite terms, called *series,* as shown in Eq. (3). The series in Eq. (3) is called *Fourier series expansion.*

How is the case where $f(x)$ is not periodic? It is considered as the case where the period L tends to the infinity. When L tends to the infinity, the intervals between the basic periods of the waves, $L/2, L/3, L/4, \ldots, L/n, \ldots,$ become smaller and smaller, and finally the intervals disappear. It indicates that the original function cannot be expressed by a summation such as Eq. (3) since the intervals are not discrete but continuous and the waves are not countable. In this case, the Fourier transform $F(\nu)$ as in Eq. (7) is not regarded as an arrangement of peaks but a continuous function composed by connecting adjacent peaks.

Cosine wave and exponential function

From Euler's equation,

$$\exp(i\omega) = \cos\omega + i\sin\omega, \tag{9}$$

we get the following relationship between the trigonometric functions and the exponential function:

$$\cos\omega = \frac{\exp(i\omega) + exp(-i\omega)}{2}$$
$$\sin\omega = \frac{\exp(i\omega) - exp(-i\omega)}{2i}. \tag{10}$$

From Eq. (9), a cosine wave $a_1 \cos 2\pi\nu_1 x$ in the real domain is expressed using exponential functions as follows:

$$a_1 \cos 2\pi\nu_1 x = \frac{a_1}{2}\exp(i2\pi\nu_1) + \frac{a_1}{2}\exp(i2\pi(-\nu_1)), \tag{11}$$

and in its Fourier transform we get two peaks of height $a_1/2$ at frequencies ν_1 and $-\nu_1$. This shows that one cosine wave is expressed by a combination of positive and negative frequencies by Fourier transformation.

On the other hand, considering the wave of the same amplitude and frequency with phase shift θ, we get

$$\begin{aligned} & a_1\cos(2\pi\nu_1 x + \theta) \\ = & \frac{a_1}{2}\exp(i(2\pi\nu_1 + \theta)) \\ & + \frac{a_1}{2}\exp(-i(2\pi\nu_1 + \theta)) \\ = & \frac{a_1}{2}\exp(i2\pi\nu_1)\exp(i\theta) \\ & + \frac{a_1}{2}\exp(i2\pi(-\nu_1)\exp(-i\theta) \end{aligned} \tag{12}$$

In this case, the amplitudes of peaks at ν_1 and $-\nu_1$ are $\frac{a_1}{2}\exp(i\theta)$ and $\frac{a_1}{2}\exp(-i\theta)$, respectively. In the case of Eq. (12), considering the axes of complex amplitude and phase, we get that the amplitude is the same as Eq. (11), however, the additional peaks of height θ along the axis of complex phase appear at ν_1 and $-\nu_1$. This example shows that a phase shift of sinusoidal waves appears as a variation in complex phase in the frequency domain.

Reference

J.W.Goodman, *Introduction to Fourier Optics,* McGraw-Hill

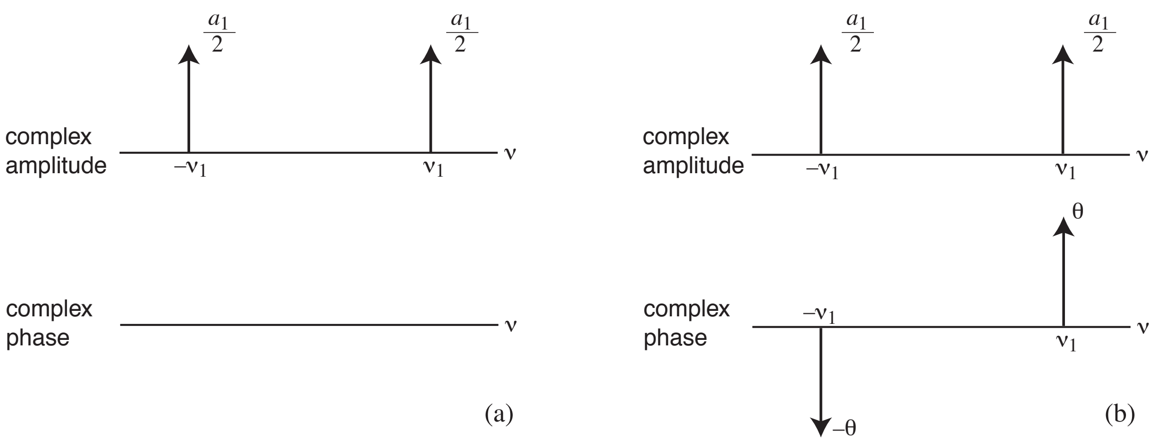

Fig. 6: Expression of phase in frequency domain. (a) $a_1 \cos 2\pi\nu_1 x$. (b) $a_1 \cos(2\pi\nu_1 x + \theta)$.