2006 Autumn semester Pattern Information Processing
Topic 1. Sampling and digital processing of images

Session 3. (2) Sampling theorem and discrete Fourier transformation

To convert a continuous distribution of brightness to a digital image, it is necessary to extract the brightness at each point arranged regularly at a period. This operation is called *sampling.* The original continuous distribution of brightness can be reconstructed if the interval is sufficiently small. The sampling theorem gives the maximum period for lossless reconstruction. Fourier transformation defined for sampled digital images is called *discrete Fourier transformation,* which is used for digital computation of Fourier transformation. The sampling theorem and the discrete Fourier transformation will be explained in this session.

Sampling and sampling theorem

Assume an image as a one-dimensional function for simplicity. Let the brightness at position x, i. e. pixel value of pixel at x, be $f(x)$. Extracting the brightness at points arranged regularly at a period is called sampling, as shown in Fig. 1.

Let us suppose a function composed by infinite numbers of Dirac's delta functions arranged at an interval T, as shown in Fig. 2. This is called *comb function,* defined as follows:

$$\text{comb}_T(x) = \sum_{n=-\infty}^{\infty} \delta(x - nT). \qquad (1)$$

A sampled digital image from $f(x)$, denoted $f_T(x)$, is expressed as $f(x)$ multiplied by the comb function $\text{comb}_T(x)$, i. e.

$$f_T(x) = f(x)\text{comb}_T(x). \qquad (2)$$

Now we consider the Fourier transformation of $f_T(x)$ to find the frequency range of sampled image $f_T(x)$. We apply the following theorem on the Fourier transformation of the product of two functions:

$$FT[f(x)g(x)](\nu) = FT[f(x)](\nu) * FT[g(x)](\nu) \qquad (3)$$

where $FT[f(x)]$ denotes the Fourier transform of $f(x)$, and the symbol $*$ denotes convolution, defined as follows:

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(y)g(t-y)dy. \qquad (4)$$

The theorem of Eq. (3) states that *the Fourier transform of the product of two functions is equal to the convolution of the Fourier transforms of the two functions* (see Appendix 1 for the proof).

We get from this theorem that the Fourier transformation of Eq. (2) is

$$FT[f_T(x)](\nu) = FT[f(x)](\nu) * FT[\text{comb}_T(x)](\nu). \qquad (5)$$

The first term of the right side of Eq. (5) is the Fourier transform of the original function $f(x)$. The second term is the Fourier transform of comb function. We get from a theorem that

$$FT[\text{comb}_T(x)](\nu) = \frac{1}{T}\text{comb}_{1/T}(\nu). \qquad (6)$$

(see Appendix 2 for the outline of the proof.) This relationship states that the Fourier transform of a comb function is also a comb function, and the period of the original comb function and that of the transformed comb function in the frequency domain are in invert proportion. Consequently, we get

$$FT[f_T(x)](\nu) = \frac{1}{T}\{FT[f(x)](\nu) * \text{comb}_{1/T}(\nu)\}. \qquad (7)$$

What is "convolution with comb function?" We explain it by starting from "convolution with delta function." From Eq. (4), we get

$$f(t) * \delta(t) = \int_{-\infty}^{\infty} f(y)\delta(t-y)dy. \qquad (8)$$

At the right side of Eq. (8), y varies from $-\infty$ to ∞. Since $\delta(t-y) = 0$ except $t = y$, the contribution of $\delta(t-y)$ to the integral is zero in this case . Thus we get

$$\begin{aligned} f(t) * \delta(t) &= \int_{-\infty}^{\infty} f(y)\delta(t-y)dy \\ &= \int_{-\infty}^{\infty} f(y)\delta(t-t)dy \\ &= f(t)\int_{-\infty}^{\infty} \delta(0)dy = f(t), \qquad (9) \end{aligned}$$

i. e. *the convolution of a function and the delta function is equal to the original function itself.*

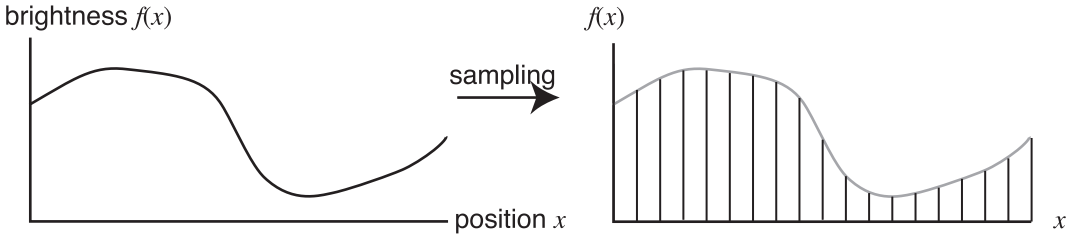

Fig. 1: Sampling.

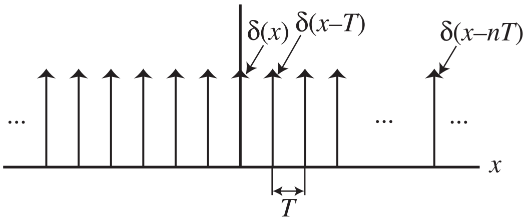

Fig. 2: Comb function.

Since a comb function is a sequence of the delta functions arrraged at a constant period, the convolution of a function and a comb function is an arrangement of shifted duplications of the original function at a constant period. Consequently, Eq. (7) states that the Fourier transform of $f_T(x)$, which is the sampled version of $f_T(x)$ at period T, is an infinite sequence of shifted duplications of $FT[f_T(x)]$, the Fourier transform of the original $f(x)$, arranged at interval $1/T$. This relationship is illustrated in Fig. 3. Here ν_c is called *cutoff frequency* and indicates the highest frequency contained in the original $f(x)$. As explained in the previous session, if $f(x)$ is a real function, the components of $FT[f(x)]$ lie in the range between $-\nu_c$ and ν_c, since $FT[f(x)](-\nu) \neq 0$ if $FT[f(x)](\nu) \neq 0$.

If the period of comb function in the frequency domain is sufficiently large, as shown in Fig. 4(a), adjacent $FT[f(x)]$'s do not overlap. In this case, the Fourier transform of the original function, $FT[f(x)]$, can be separated and extracted, i. e. no information of the brightness distribution of the original image is lost by the sampling. However, if the interval of the comb functions in the frequency domain is small, as shown in Fig. 4(b), adjacent $FT[f(x)]$'s overlap. In this case, the original $FT[f(x)]$ cannot be separated and a faulty function will be extracted. This effect is called *aliasing*.

Since the support of $FT[f(x)]$ is in the range between $-\nu_c$ and ν_c, the period has to be at least $2\nu_c$ for avoiding overlaps of $FT[f(x)]$'s. Since T is a sampling period, $1/T$ denotes the number of samples per unit length, i. e. sampling rate. Consequently, *the original brightness distribution can be reconstructed by a sampled digital image if the sampling rate is more than twice the maximum frequency contained in the original distribution.* This theorem is called *sampling theorem.*

For example, the music compact disc is recorded digitally at sampling rate 44.1kHz, i. e. the signal is sampled 44100 times per second. Thus the maximum frequency that the music CD can correctly reproduce is 22.05kHz. It means that the filtering for cutting off the frequency range higher than 22.05kHz at the recording process is required for avoiding an aliasing.

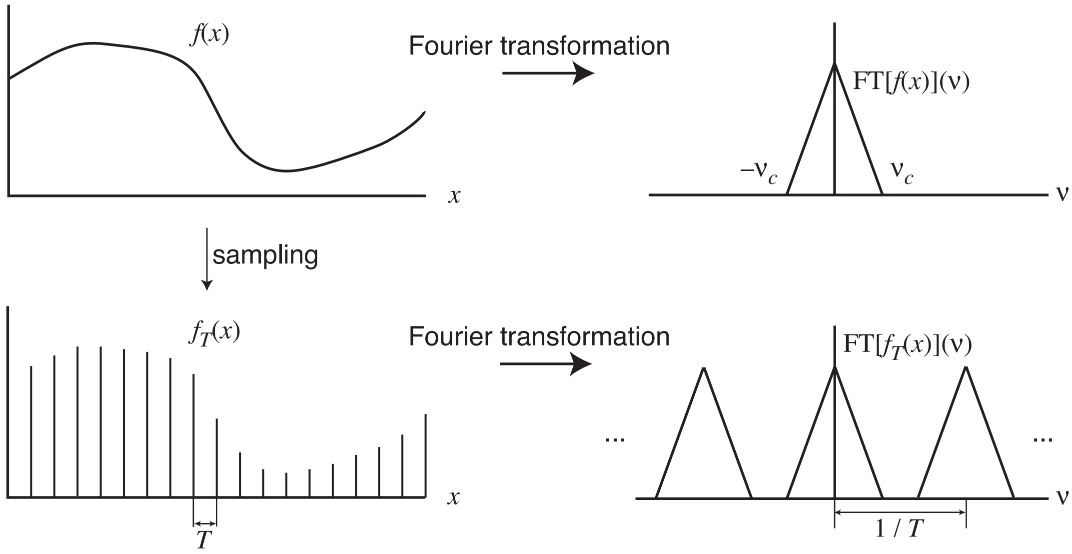

Fig. 3: Sampling and Fourier transformation.

Discrete Fourier transformation

It is often easier to treat continuous functions than discrete ones in mathematics; however, the digital computer can handle discrete functions only. In this section, we consider the Fourier transformation of discrete functions.

As defined in the previous session, the Fourier transform of the sampled function $f_T(x)$ is as follows:

$$\begin{aligned} & FT[f_T(x)](\nu) \\ = & FT[f(x)\mathrm{comb}_T(x)](\nu) \\ = & \int_{-\infty}^{\infty} f(x)\mathrm{comb}_T(x)\exp(-i2\pi\nu x)dx. \end{aligned} \quad (10)$$

By the definition of comb function in Eq. (1), we get from Eq. (10) that

$$\begin{aligned} & FT[f_T(x)](\nu) \\ = & \int_{-\infty}^{\infty} f(x)\sum_{n=-\infty}^{\infty}\delta(x-nT)\exp(-i2\pi\nu x)dx \\ = & \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\{f(x)\exp(-i2\pi\nu x)\}\delta(x-nT)dx \end{aligned} \quad (11)$$

Similarly to Eq. (9), the integral in Eq. (11) is rewritten as follows:

$$\begin{aligned} & \int_{-\infty}^{\infty}\{f(x)\exp(-i2\pi\nu x)\}\delta(x-nT)dx \\ = & \int_{-\infty}^{\infty}\{f(nT)\exp(-i2\pi\nu x)\}\delta(nT-nT)dx \\ = & f(nT)\exp(-i2\pi\nu nT). \end{aligned} \quad (12)$$

Thus we get from Eq. (11) that

$$FT[f_T(x)](\nu) = \sum_{n=-\infty}^{\infty} f(nT)\exp(-i2\pi\nu nT). \quad (13)$$

Since $f_T(x)$ is sampled from $f(x)$, $f(nT) = f_T(nT)$. Thus we get from Eq. (13) that

$$FT[f_T(x)](\nu) = \sum_{n=-\infty}^{\infty} f_T(nT)\exp(-i2\pi\nu nT). \quad (14)$$

$f_T(x)$ has been regarded as the sampled version of $f(x)$ so far; Now we consider the situation that only $f_T(x)$ is given and the original $f(x)$ is unknown. In this case $f_T(x)$ is simply a sequence of numbers. We also rename the sampling period T([meters]) to 1([unit length] or [step]), since one step of the sequence corresponds to length T of $f_T(x)$. Then we get the sequence $u(n)$ as follows:

$$u(n) = f_T(nT). \quad (15)$$

Consequently, one step of the sequence $u(n)$ corresponds to length T([meters]) in $f_T(x)$. Then rewriting

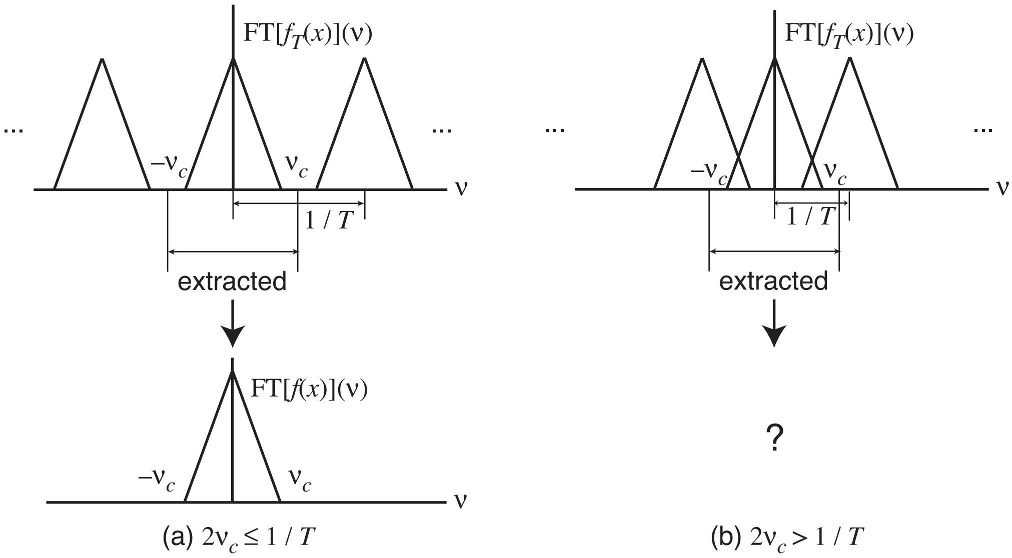

Fig. 4: Sampling theorem.

Eq. (14), we define $\tilde{U}(\nu)$ as follows:

$$\tilde{U}(\nu) = \sum_{-\infty}^{\infty} u(n) \exp(-i2\pi\nu n). \quad (16)$$

$\tilde{U}(\nu)$ is regarded as *the Fourier transform of sequence u(n)*[1]. As explained above, the Fourier transform of a sampled function is periodic in the frequency domain. Since the sampling period is 1([unit length]), the period of the Fourier transform is 1(1/[unit length]). We consider only one period in the frequency domain in the followings.

A continuous function is obtained in the frequency domain by the Fourier transformation of a sequence, although the original function is discrete in the real domain. It is inconvenient for digital computation. Thus we take N samples from $\tilde{U}(\nu)$, i. e. the sampling rate is $1/N$ (1 / [unit length]). We get from this sampling the following:

$$\tilde{U}(\frac{k}{N}) = \sum_{-\infty}^{\infty} u(n) \exp(-i2\pi\frac{k}{N}n) \;\; (k = 0, 1, \ldots, N-1). \quad (17)$$

What happens in the real domain by this frequency domain sampling? The answer is obtained by the reverse operation of the sampling theorem. The operation to obtain the real domain function $f(x)$ from the frequency domain function $F(\nu)$ is called *inverse Fourier transformation.* As explained in the previous session, since the component of frequency ν in the frequency domain corresponds to the coefficient of $\exp(i2\pi\nu x)$, the inverse Fourier transformation is defined as follows [2]:

$$FT^{-1}[F(\nu)](x) = \int_{-\infty}^{\infty} F(\nu) \exp(i2\pi\nu x) d\nu = f(x) \quad (18)$$

This is very similar to the original Fourier transformation. Thus the discussion about the sampling theorem also holds for the inverse Fourier transformation, i. e. sampling in the frequency domain by interval $1/N$ yields the periodic duplications of the original function at period N in the real domain. Thus the sampling of the Fourier transform of a sequence at period $1/N$ in the frequency domain corresponds to the Fourier transform of the N-point sequence extended to the periodic function by the infinite duplications of the sequence in the real domain. If the original sequence contains exactly N numbers, the discrete Fourier transform of the original sequence

[1] This operation is also refered as *the discrete time Fourrier transformation (DTFT) of function u(n) of discrete time n*, since the axis of real domain is often the time for one dimensional functions.

[2] The right side of this equation is divided by N for normalization in some textbooks. This is explained later in the section of the unitary transformation.

is properly obtained by the sampling at period $1/N$ in the frequency domain, in the sense of obtaining the Fourier transform of the periodic extension of the sequence.

Now we assume that $u(n)$ is a finite sequence of N numbers, i. e. $u(n) = 0$ except $n = 0, 1, \ldots, N?1$. Then we get from Eq. (17) that

$$\tilde{U}(\frac{k}{N}) = \sum_{n=0}^{N-1} u(n) \exp(-i2\pi \frac{k}{N} n) \ \ (k = 0, 1, \ldots, N-1). \tag{19}$$

Renaming the left side to $U(k)$, we get

$$U(k) = \sum_{n=0}^{N-1} u(n) \exp(-i2\pi \frac{k}{N} n) \ \ (k = 0, 1, \ldots, N-1). \tag{20}$$

This operation is called *discrete Fourier transformation (DFT)*. The Fourier transformation perfomed by digital computers is always the discrete Fourier transformation.

As an example, we assume a one-dimensional signal $u(n)$ of $N = 256$ points sampled at period $T = 1$([milimeter]). Now we consider the period equivalent to one step of k in the frequency domain, if we get $U(k)$ from $u(n)$ by the discrete Fourier transformation. The Fourier transformation of a signal sampled at period T([meters]) yields the repetition of the Fourier transform of the original signal by period $1/T$([1/meter]) in the frequency domain. Since one step of the discrete Fourier transform is equal to one of N divisions of the period of $1/T$([1/meter]), one step corresponds to 1/ NT([1/meter]). Thus one step corresponds to $1/256$([1/ milimeter]) in this example, since $T = 1$([milimeter]) and $N = 256$([steps]). Figure 5 illustrates this relationship.

Note that $U^*(N?k) = U(k)$ for the discrete Fourier transformation of a sequence of N real numbers. It is proved as follows: Let $U^*(k)$ be the complex conjugate of $U(k)$. If $u(n)$ is a sequence of N real numbers, we get

$$\begin{aligned} U^*(N-k) &= \sum_{n=0}^{N-1} u^*(n) \exp(i2\pi \frac{N-k}{N} n) \\ &= \sum_{n=0}^{N-1} u(n) \exp(i2\pi n) \exp(i2\pi \frac{-k}{N} n). \end{aligned} \tag{21}$$

Since $\exp(i2\pi n) = 1$ for integer n, we get

$$\begin{aligned} U^*(N-k) &= \sum_{n=0}^{N-1} u(n) \exp(i2\pi \frac{-k}{N} n) \\ &= U(k). \end{aligned} \tag{22}$$

This relationship means that the highest frequency that can be expressed by the discrete Fourier transformation of N points is $N/2$, and only $N/2+1$ numbers from $U(0)$ to $U(N/2)$ are unique. It is also found that the maximum frequency corresponds to $k = N/2$, i. e. the middle of the sequence, and zero frequency corresponds to $k = 0$, i. e. the end of the sequence.

Appendix 1. Convolution and Fourier transformation

Let F and G be Fourier transforms of real-domain functions f and g, respectively. We get from Eq. (18), the definition of the inverse Fourier transformation, that

$$\begin{aligned} & f(x)g(x) \\ =& \int_{-\infty}^{\infty} F(\nu) \exp(i2\pi\nu x) d\nu \int_{-\infty}^{\infty} G(\mu) \exp(i2\pi\mu x) d\mu \\ =& \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\nu)G(\nu) d\nu \exp(i2\pi(\nu+\mu)x) d\mu. \end{aligned} \tag{23}$$

Applying the variable conversion $\nu + \mu = \xi$, we get

$$\begin{aligned} f(x)g(x) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\nu)G(\xi-\nu) d\nu \exp(i2\pi\xi x) d\xi \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [F * G](\xi) \exp(i2\pi\xi x) d\xi \\ &= FT^{-1}[F * G](x) \end{aligned} \tag{24}$$

Equation (3) is obtained by the inverse transformation of Eq. (24).

Appendix 2. Fourier transformation of comb function

From the definition of comb$T(x)$ by Eq. (1), we get that comb$T(x)$ is a periodic function of period T. A periodic function of period T is expressed by a series of sinusoidal functions whose frequency is n/T (n : integer), i. e.

$$\mathrm{comb}_T(x) = \sum_{n=-\infty}^{\infty} a_n \exp(i2\pi\frac{n}{T}x). \qquad (25)$$

The coefficient a_n at the frequency n/T is obtained by multiplication of $\exp(-i2\pi\frac{n}{T}x)$ and integration, because of the property of the orthogonal function system. Since it is a periodic function of period T, the range of integration is not $(-\infty, \infty)$ but $[-T/2, T/2]$. Multiplying $1/T$ for normalization, we get

$$\begin{aligned} a_n &= \frac{1}{T}\int_{-T/2}^{T/2} \mathrm{comb}_T(x)\exp(-i2\pi\frac{n}{T}x)dx \\ &= \frac{1}{T}\int_{-T/2}^{T/2} \delta(x)\exp(-i2\pi\frac{n}{T}x)dx \\ &= \frac{1}{T}\exp(-i2\pi\frac{n}{T}\cdot 0) = \frac{1}{T}, \end{aligned} \qquad (26)$$

and then we get [3]

$$\mathrm{comb}_T(x) = \frac{1}{T}\sum_{n=-\infty}^{\infty} \exp(i2\pi\frac{n}{T}x). \qquad (27)$$

Thus its Fourier transform is as follows:

$$FT[\mathrm{comb}_T(x)](\nu) = \frac{1}{T}\sum_{n=-\infty}^{\infty} FT[\exp(i2\pi\frac{n}{T}x)](\nu). \qquad (28)$$

As explained in the previous session, since we get peaks at $\nu = n/T$ in the Fourier transform of $\exp(i2\pi\frac{n}{T}x)$, we get from Eq. (28) that

$$\begin{aligned} FT[\mathrm{comb}_T(x)](\nu) &= \frac{1}{T}\sum_{n=-\infty}^{\infty} \delta(\nu - \frac{n}{T}) \\ &= \frac{1}{T}\mathrm{comb}_{1/T}(\nu). \end{aligned} \qquad (29)$$

Reference

H.P.Hsu, *Applied Fourier Analysis,* ISBN0-15-601609-5
（佐藤平八訳，フーリエ解析，森北出版 ISBN4-627-93010-0）

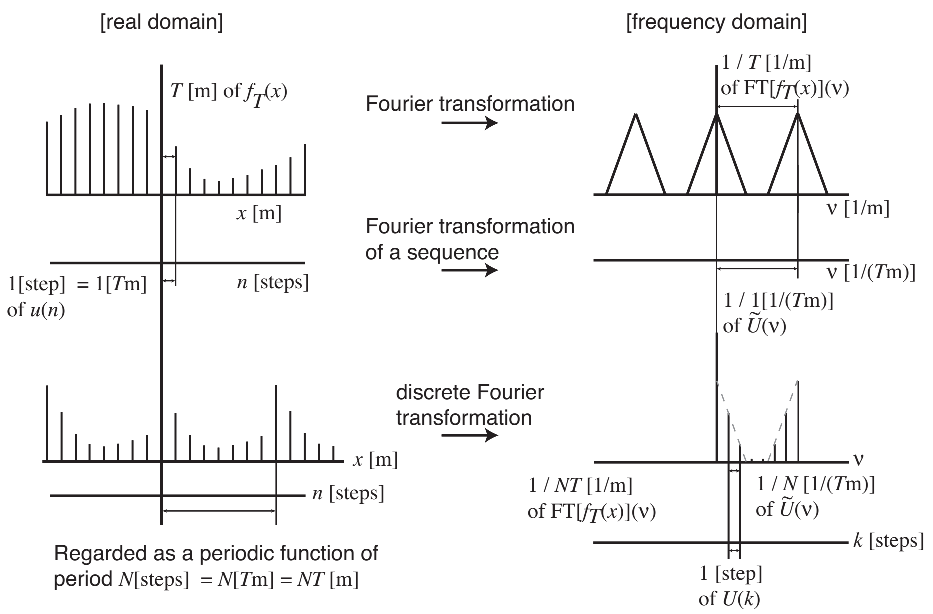

Fig. 5: Discrete Fourier transformation.

[3] This operation is called Fourier series expansion. See Hsu's book in Reference.