2006 Autumn semester Pattern Information Processing
Topic 5. Computed Tomography - Image reconstruction from projection

Session 13. (1) Radon transformation and projection theorem

The final topic of this course treats the computed tomography (CT), which can take cross-section images of human body. This topic explains the principle of CT by the following two sessions: (1) Radon transformation and the projection theorem for formalization of projection, and (2) Image reconstruction from projections.

What is CT scanner?

CT scanner is an imaging equipment which reconstructs a cross-section image of a 3-D object from fluoroscopic images taken from around the object from various angles. The fluoroscopic images are obtained by X-ray imaging or nuclear magnetic resonance (NMR)[1] imaging methods. One of the newest scanners can obtain cross-section images in real time, and can obtain a 3-D image of the whole body by helical scan around the body. Another scanner which is set around a surgical operation table and obtains cross-sections or 3-D images during the operation has also been developed. Besides the medical equipments, industrial CT scanners for nondestructive testing of the inside of metal devices or foods have been already in practical use.[2]

Fluoroscopic images are obtained as illustrtated in Fig. 1. Practical scanners obtain the images by the "fan-beam method," as shown in Fig. 1 (b). The Xrays are emitted from a point source and captured by the detectors on an arc at the opposite position of the source. In this course, however, it is assumed that the X-rays are emitted parallelly for simplicity of explanation, as shown in Fig. 1 (a). Since the source and detectors rotate as preserving their relative positions, (a) and (b) are equivalent and X-rays from all directions pass at every points within the object.

The X-ray reaching a position of the detector is absorbed by the object at each point on the X-ray pass. Thus the intensity of the X-ray reaching a position of the detector is proportional to the integral of the 2-D transparence distribution of the object along the pass. The generation of cross-section images by the CT scanner is the reconstruction of the 2-D transparence distribution functions from the set of 1-D functions obtained by integrals along the lines of various directions. These integrals are called *projections* of the 2-D distribution and our problem is called *reconstruction of distribution from projection.*

Radon transfromation

The following theorem by Radon states that image reconstruction from projection is possible:

> The value of a 2-D function at an arbitrary point is uniquely obtained by the integrals along the lines of all directions passing the point.

This theorem guarantees that a 2-D object (equivalent to a transparence distribution) is reconstructed from projections obtained by the rotational scanning shown in the previous section.

The Radon transformation shows the relationship between the 2-D object and the projections. Let us consider a coordinate system shown in Fig. 2. The function $g(s, \theta)$ is a projection of $f(x, y)$ on the axis s of θ direction. The function $g(s, \theta)$ is obtained by the integration along the line whose normal vector is in θ direction. The value $g(0, \theta)$ is defined that it is obtained by the integration along the line passing the origin of (x, y)-coordinate.

Since the points on the line whose normal vector is in θ direction and passing the origin of (x, y)-

[1]Although this type of CT had been called NMR-CT, the name MRI (magnetic resonace imaging) is currently popular since the word "nuclear," reminding of nuclear weapons, has been avoided.

[2]`http://www.toshiba-itc.com/cat/en/prod01.html`

coordinate satisfy

$$\frac{y}{x} = \tan(\theta + \frac{\pi}{2}) = \frac{-\cos\theta}{\sin\theta}, \tag{1}$$

we get

$$x\cos\theta + y\sin\theta = 0. \tag{2}$$

The integration along the line whose normal vector is in θ direction and that passes through the origin of (x, y)-coordinate means the integration of $f(x, y)$ only at the points satisfying Eq. (2), $g(0, \theta)$ is expressed using the δ-function as follows:

$$g(0, \theta) = \iint_{-\infty}^{\infty} f(x, y)\delta(x\cos\theta + y\sin\theta)dxdy. \tag{3}$$

Similarly, it follows from Eq. (2) that the line whose normal vector is in θ direction and whose distance from the origin is s satisfy the following equation:

$$(x - s\cos\theta)\cos\theta + (y - s\sin\theta)\sin\theta = 0, \tag{4}$$

i. e.

$$x\cos\theta + y\sin\theta - s = 0, \tag{5}$$

since this line is obtained by moving the line passing through the origin by $s\cos\theta$ in x direction and $s\sin\theta$ in y direction. Thus similarly to Eq. (3) we get

$$g(s, \theta) = \iint_{-\infty}^{\infty} f(x, y)\delta(x\cos\theta + y\sin\theta - s)dxdy. \tag{6}$$

The Eq. (6) is called Radon transformation from the 2-D distribution $f(x, y)$ to the projection $g(s, \theta)$.

Ray-sum

Although the Radon transformation expresses the projection by the 2-D integral on the x, y-coordinate, the projection is more naturally expressed by an integral of one variable since it is a line integral. Let us consider rewriting Eq. (6) to an integral of one variable.

Since the (s, u)-coordinate along the direction of projection is obtained by rotating the (x, y)-coordinate by θ, the relationship between two directions is expressed as follows:

$$\begin{pmatrix} s \\ u \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{7}$$

Thus we get the following relationship between (s, u) and (x, y):

$$\begin{cases} s = x\cos\theta + y\sin\theta \\ u = -x\sin\theta + y\cos\theta, \end{cases} \tag{8}$$

$$\begin{cases} x = s\cos\theta - u\sin\theta \\ y = s\sin\theta + u\cos\theta. \end{cases} \tag{9}$$

Substituting Eq. (9) into Eq. (6), it follows that the argument of the δ-function is

$$\begin{aligned} & x\cos\theta + y\sin\theta - s \\ = \; & (s\cos\theta - u\sin\theta)\cos\theta + (s\sin\theta + u\cos\theta)\sin\theta - s \\ = \; & s(\cos^2\theta + \sin^2\theta) - u\sin\theta\cos\theta + u\sin\theta\cos\theta - s \\ = \; & 0. \end{aligned} \tag{10}$$

Since the translation from the (x, y)-coordinate to the (s, u)-coordinate yields no expansion or shrinkage, we get $dxdy = dsdu$. Thus we get from Eq. (6)

$$g(s, \theta) = \iint_{-\infty}^{\infty} f(s\cos\theta - u\sin\theta, s\sin\theta + u\cos\theta)\delta(0)dsdu. \tag{11}$$

Since the δ-function in Eq. (6) is a function of variable s, we get

$$\int_{-\infty}^{\infty} \delta(0)ds = 1. \tag{12}$$

It follows from the above that transformation $g(s, \theta)$ in Eq. (6) is translated integral of one variable u,

$$g(s, \theta) = \int_{-\infty}^{\infty} f(s\cos\theta - u\sin\theta, s\sin\theta + u\cos\theta)du. \tag{13}$$

This equation expresses the sum of $f(x, y)$ along the X-ray path whose distance from the origin is s and whose normal vector is in θ direction. This sum, $g(s, \theta)$ is called *ray-sum.*

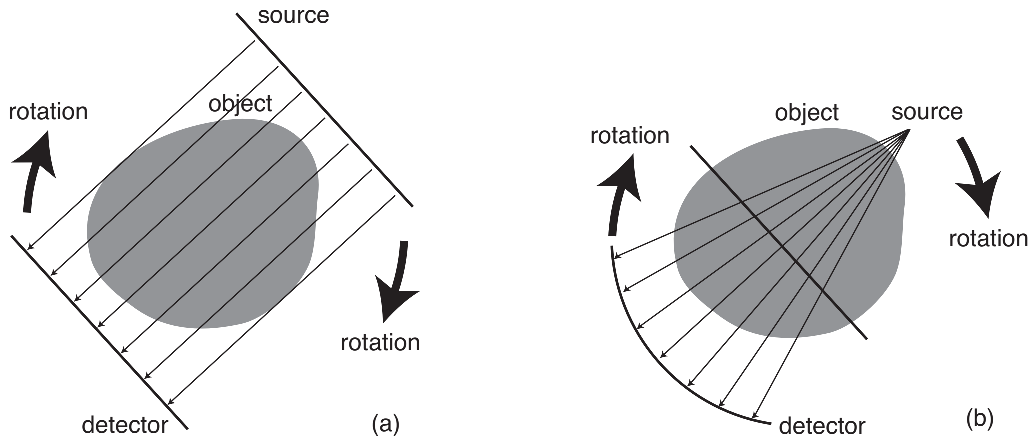

Fig. 1: Obtaining fluoroscopic images by CT scanner. (a) parallel emission. (b) fan-beam emission.

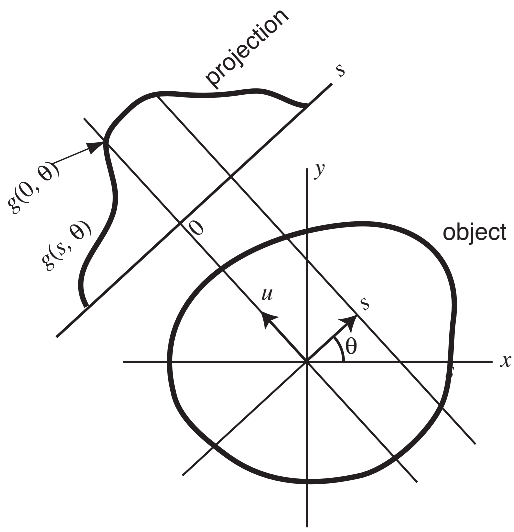

Fig. 2: Radon transformation.

Projection theorem

The image reconstruction from projection is equivalent to the inverse Radon transformation, i. e. obtaining $f(x, y)$ from a given $g(s, \theta)$ for $0 \leq \theta < \pi$.[3] An important key for solving this problem is *projection theorem,* explained in the following.

The projection theorem states that

> the one-dimensional Fourier transform of the Radon transform $g(s, \theta)$ for variable s, denoted $G_\theta(\xi)$, and
> the cross-section of the two-dimensional Fourier transform of the object $f(x, y)$, sliced by the plane at θ with the f_x-coordinate and perpendicular to the (f_x, f_y)-plane, denoted $F(f_x, f_y)$,
> are identical, i. e.

$$G_\theta(\xi) = F(\xi \cos\theta, \xi \sin\theta). \tag{14}$$

Proof of the theorem is as follows: The one-dimensional Fourier transform of the Radon transform $g(s, \theta)$ for variable s, denoted $G_\theta(\xi)$, is ex-

[3]Note that the range is *not* $0 \leq \theta < 2\pi$.

pressed as follows:

$$G_\theta(\xi) = \int_{-\infty}^{\infty} g(s,\theta)\exp(-i2\pi\xi s)ds. \qquad (15)$$

Substituting the definition of ray-sum, Eq. (13), into Eq. (15), we get

$$\begin{aligned} G_\theta(\xi) &= \iint_{-\infty}^{\infty} f(s\cos\theta - u\sin\theta, s\sin\theta + u\cos\theta) \\ &\quad \times\exp(-i2\pi\xi s)dsdu. \qquad (16)\end{aligned}$$

We get from the substitution of the variables (x, y) into (s, u), using the relationship $dxdy = dsdu$ mentioned above,

$$\begin{aligned} G_\theta(\xi) &= \iint_{-\infty}^{\infty} f(x,y) \\ &\quad \exp(-i2\pi\xi(x\cos\theta + y\sin\theta))dxdy \\ &= \iint_{-\infty}^{\infty} f(x,y) \\ &\quad \exp(-i2\pi((\xi\cos\theta)x + (\xi\sin\theta)y))dxdy \\ &= F(\xi\cos\theta, \xi\sin\theta). \qquad (17)\end{aligned}$$

Reconstruction by Fourier transformation method

The projection theorem indicates that the projection at an angle θ yields one cross-section of $F(f_x, f_y)$, the Fourier transform of the original object. Thus the projections for all θ yields the whole profile of $F(f_x, f_y)$. The inverse Fourier transformation of $F(f_x, f_y)$ obtained above yields the full reconstruction of $f(x, y)$. This reconstruction method is called *Fourier transformation method.*

Although this method is theoretically the simplest of various reconstruction methods, it is practically not popular by the following reason.

Obtaining fluoroscopic images for *all* θ is practically impossible; they are obtained at an interval of θ. The Fourier transformation of $g(s, \theta)$ is calculated practically by computers using the discrete Fourier transformation with sampled s. Thus $F(f_x, f_y)$ is obtained only at discrete points located radially on (f_x, f_y)- plane. The discrete inverse Fourier transformation of $F(f_x, f_y)$ requires $F(f_x, f_y)$ at square lattice points. Since the radially located points and the lattice points are not generally synchronized, the values of $F(f_x, f_y)$ at the lattice points have to be estimated from the values at the radially located points by some interpolation. The error by the interpolation in the frequency domain can yield an artifact, which is a noise not existing in the original image but caused by the processing, spread over the whole image. The artifact causes a severe misjudgment in image-aided medical diagnosis, since such diagnosis should find an object that should not be normally observed, for example a tumor.

References

A. K. Jain, Fundamentals of Digital Image Processing.

Imaginis – How Does CT Work?
(http://www.imaginis.com/ct-scan/how_ct.asp)

Computed Tomography (CT) - Head
(http://www.radiologyinfo.org/content/ct_of_the_head.htm)

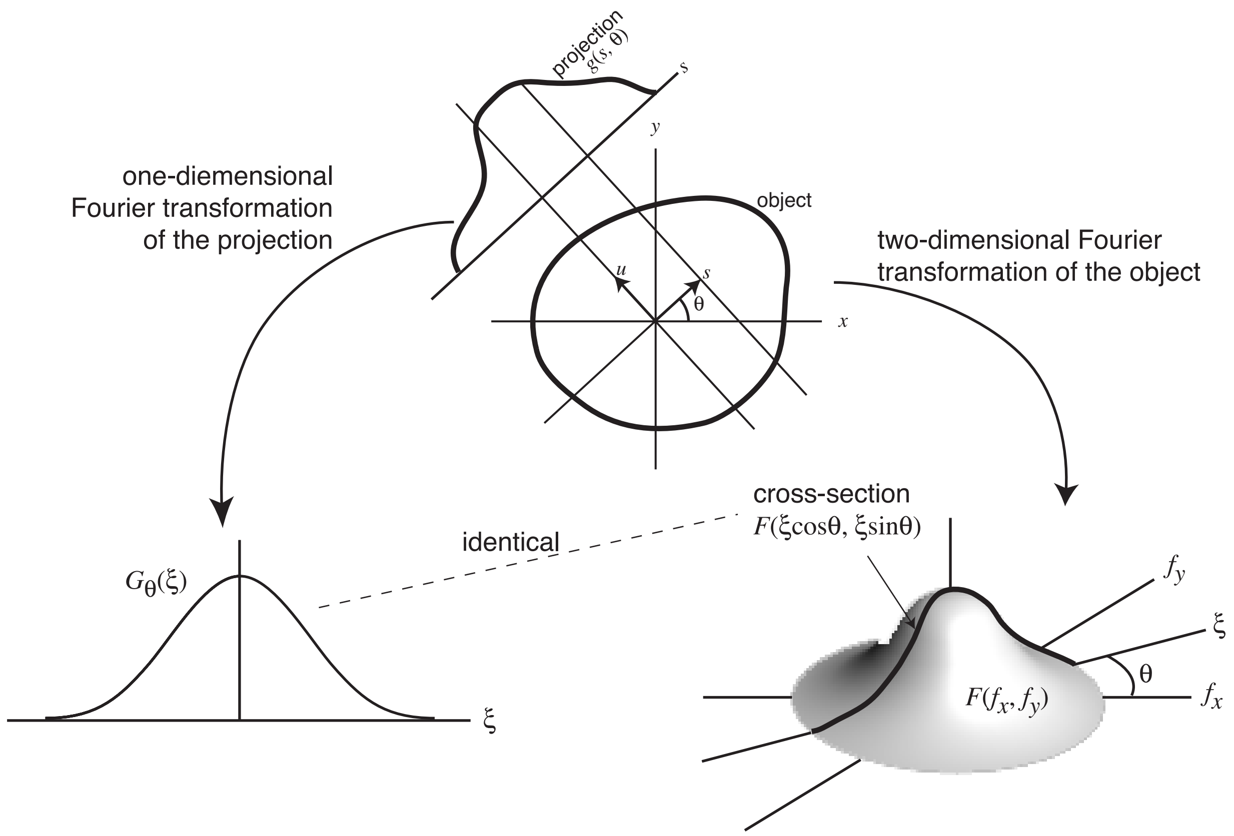

Fig. 3: Projection theorem.