2006 Autumn semester Pattern Information Processing
Topic 5. Computed Tomography - Image reconstruction from projection

Session 14. (2) Image reconstruction from projections

In the previous session, we explained the projection theorem and the defect of the reconstruction by the Fourier transformation method, which is a direct application of the projection theorem. We explain a more practical method derived from the simple back-projection in this session.

Back-projection method

Let us consider a simpler reconstruction method. To reconstruct $f(x, y)$, which is the absorbance at point (x, y), we consider the summation of projections passing through (x, y) for all θ. Since these projections are line integrals through $f(x, y)$, $f(x, y)$ is duplicated and enhanced in the summation. Thus $f(x, y)$ is reconstructed by this summation although it contains blur by absorbances at other points included in the projections. This reconstruction method is called *back-projection method.* Let us consider whether this method really achieves the reconstruction.

A part of Radon transform $g(s, \theta)$ projected on the axis of angle θ and passing through (x, y) is $g(x \cos\theta + y \sin\theta, \theta)$ because

$$s = x \cos\theta + y \sin\theta, \tag{1}$$

explained in the previous session. The summation of $g(x \cos\theta + y \sin\theta, \theta)$ for all θ yields the reconstructed image by the back-projection method, denoted $b(x, y)$, i. e.

$$b(x, y) = \int_0^{\pi} g(x \cos\theta + y \sin\theta, \theta) d\theta. \tag{2}$$

Substituting the definition of the Radon transformation (Eq. (6) in the previous session),

$$g(s, \theta) = \iint_{-\infty}^{\infty} f(x, y)\delta(x \cos\theta + y \sin\theta - s) dx dy, \tag{3}$$

and Eq (1) into Eq. (2), we get

$$\begin{aligned} b(x, y) &= \int_0^{\pi} \left[\iint_{-\infty}^{\infty} f(x', y')\delta(x' \cos\theta + y' \sin\theta - (x \cos\theta + y \sin\theta)) dx' dy' \right] d\theta \\ &= \iint_{-\infty}^{\infty} f(x', y') \left[\int_0^{\pi} \delta((x' - x) \cos\theta + (y' - y) \sin\theta)) d\theta \right] dx' dy'. \end{aligned} \tag{4}$$

We employ here the theorem that

$$\delta[h(\theta)] = \sum_k \frac{1}{|h'(\theta_k)|} \delta[\theta - \theta_k] \tag{5}$$

if a function $h(\theta) = 0$ for a finite number of $\theta = \theta_k$ only (proof is omitted). For the argument of the δ-function in Eq. (4), it follows that

$$\begin{gathered} (x' - x) \cos\theta + (y' - y) \sin\theta) = \sqrt{(x' - x)^2 + (y' - y)^2} \sin(\theta + \alpha), \\ \alpha = \cos^{-1} \frac{y' - y}{\sqrt{(x' - x)^2 + (y' - y)^2}} = \sin^{-1} \frac{x' - x}{\sqrt{(x' - x)^2 + (y' - y)^2}} \end{gathered} \tag{6}$$

and the argument is zero, if and only if $\theta = \pi - \alpha$ for $0 \leq \theta < \pi$. Thus it follows from the δ-function in Eq. (4) that

$$\delta((x' - x) \cos\theta + (y' - y) \sin\theta)) = \frac{1}{\left| \sqrt{(x' - x)^2 + (y' - y)^2} \cos(\pi) \right|} \delta(\theta - (\pi - \alpha)), \tag{7}$$

and then we get from Eq. (4) that

$$\begin{aligned} b(x,y) &= \iint_{-\infty}^{\infty} f(x',y') \left[\frac{1}{\sqrt{(x'-x)^2 + (y'-y)^2}} \right] dx'dy' \\ &= f(x,y) * \left[\frac{1}{\sqrt{x^2+y^2}} \right]. \end{aligned} \tag{8}$$

The symbol $*$ denotes the convolution. Consequently, $b(x,y)$, the reconstructed image by the back-projection method, is obtained by blurring $f(x,y)$ by convoluting $1/\sqrt{x^2+y^2}$.

This "reconstructed image" is highly blurred and not the real reconstruction. However, the Fourier transformation of Eq. (S14backproj3) yields

$$\begin{aligned} FT[b(x,y)] &= FT[f(x,y)] \times FT\left[\frac{1}{\sqrt{x^2+y^2}} \right] \\ \text{thus } FT[f(x,y)] &= FT[b(x,y)]/FT\left[\frac{1}{\sqrt{x^2+y^2}} \right], \end{aligned} \tag{9}$$

since the Fourier transform of the convolution of two functions equals to the product of the Fourier transforms of the two functions. Employing

$$FT\left[\frac{1}{\sqrt{x^2+y^2}} \right] = \frac{1}{\sqrt{f_x^2+f_y^2}} \tag{10}$$

(proof is omitted), Eq. (9) is rewritten to

$$FT[f(x,y)] = \sqrt{f_x^2+f_y^2} \times FT[b(x,y)]. \tag{11}$$

Equation (11) yields the Fourier transform of the original object $f(x,y)$. This deblurring is called *inverse filtering,* and this kind of the inverse operation of convolution is called *deconvolution*.

This method has the following two problems: 1) $FT[b(x,y)]$ should be calculated within an area much broader than the support of $f(x,y)$, since the back-projection $b(x,y)$ is spread by blurring $f(x,y)$. 2) $f(x,y)$ is positive at every (x,y) since it is a distribution of absorbance. However, from Eq. (11), $FT[f(x,y)] = 0$ when $f_x = f_y = 0$. It means that the DC component of $f(x,y)$ is zero and negative values should appear in $f(x,y)$. This is a contradiction. The reason is that $FT[b(x,y)]$ diverges at $f_x = f_y = 0$ and no information on $f(x,y)$ is obtained there.

Filter back-projection method

Although the back-projection method cannot yield a good reconstruction as explained in the previous section, a practical reconstruction method is derived from the back-projection method using the projection theorem.

We rewrite $FT[f(x,y)]$ to $F(f_x,f_y)$. Since $f(x,y)$ is obtained by the inverse Fourier transformation of $F(f_x,f_y)$, we get

$$f(x,y) = \iint_{-\infty}^{\infty} F(f_x,f_y) \exp(i2\pi(f_x x + f_y y)) df_x df_y. \tag{12}$$

Converting this into the polar coordinate (ξ,θ) using the relationship $f_x = \xi\cos\theta$ and $f_y = \xi\sin\theta$ as shown in Fig. 1, we get

$$f(x,y) = \int_0^{2\pi} \int_0^{\infty} F(\xi\cos\theta, \xi\sin\theta) \exp(i2\pi\xi(x\cos\theta + y\sin\theta))\xi d\xi d\theta. \tag{13}$$

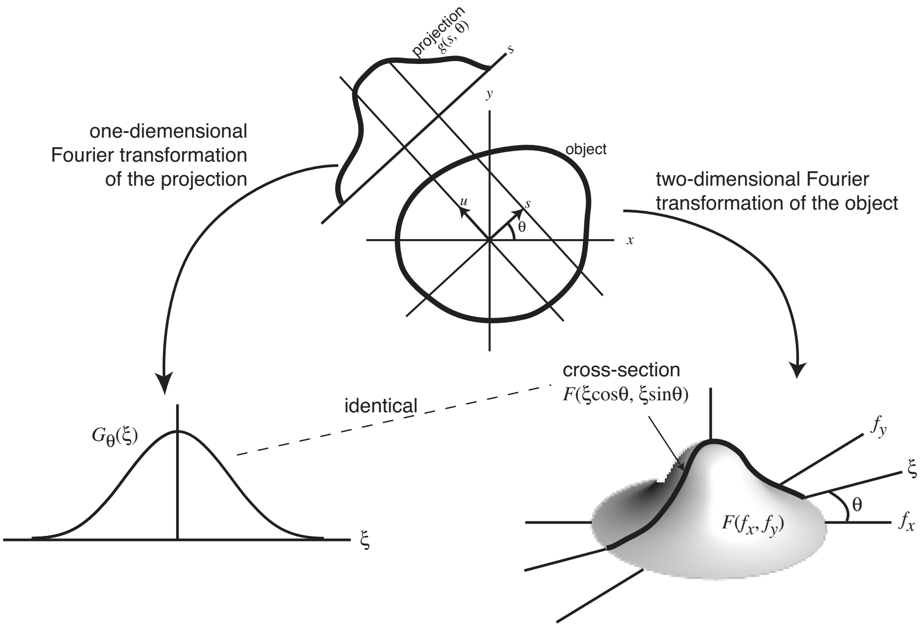

Fig. 1: Projection theorem.

The projection theorem states that

$$G_\theta(\xi) = F(\xi\cos\theta, \xi\sin\theta), \tag{14}$$

where $G_\theta(\xi)$ is the one dimensional Fourier transform of the Radon transform $g(s,\theta)$ with respect to s, as shown in Eq. (14) in the previous session. It follows that

$$f(x,y) = \int_0^{2\pi}\int_0^{\infty} G_\theta(\xi)\exp(i2\pi\xi(x\cos\theta + y\sin\theta))\xi d\xi d\theta. \tag{15}$$

Rewriting the interval of integral on ξ to $(-\infty,\infty)$ and that on θ to $(0,\pi)$, we get

$$f(x,y) = \int_0^{\pi}\int_{-\infty}^{\infty} G_\theta(\xi)\exp(i2\pi\xi(x\cos\theta + y\sin\theta))|\xi| d\xi d\theta, \tag{16}$$

and employing Eq. (1), we get

$$f(x,y) = \int_0^{\pi}\left[\int_{-\infty}^{\infty} |\xi| G_\theta(\xi)\exp(i2\pi s\xi)d\xi\right]d\theta. \tag{17}$$

The integral within { } is the inverse Fourier transform of a function $|\xi|G_\theta(\xi)$.

Defining

$$\hat{g}(s,\theta) = \hat{g}(x\cos\theta + y\sin\theta,\theta) \equiv \int_{-\infty}^{\infty} |\xi| G_\theta(\xi)\exp(i2\pi s\xi)d\xi, \tag{18}$$

we get

$$f(x,y) = \int_0^{\pi}\hat{g}(x\cos\theta + y\sin\theta,\theta)d\theta. \tag{19}$$

This is in the same form of the back-projection in Eq. (2). Consequently, Eq. (19) states that the original object $f(x,y)$ is obtained by applying the filter that multiplies $|\xi|$ to the Radon transforms and then performing the backprojection. This method is called *filter back-projection method.* This method performs the back-projection

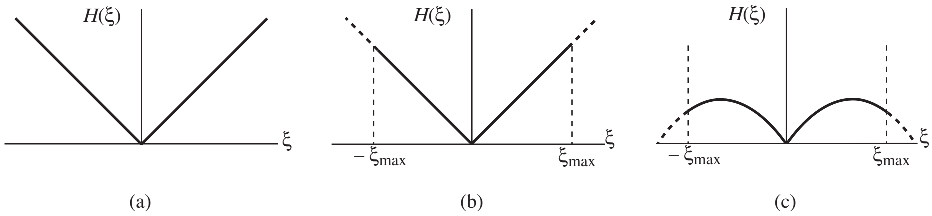

Fig. 2: Filter functions. (a) Original filter function ($|\xi|$). (b) Ram-Lak filter. (c) Shepp-Logan filter.

after applying the filter, contrarily to the back-projection with deconvolution, which is explained in the previous section, which applies the filtering after the back-projection.

This method does not require the inverse Fourier transformation of the spread blurred image, since the Fourier transformation is applied to the projections only. Although this method requires an interpolation between the polar coordinate to the Cartesian coordinate similarly to the Fourier transformation method, no artifact spread over the whole real domain is occurred, since this method carries out the interpolation in the real domain contrarily to the Fourier transformation method. Since the filtering can be applied for each θ independently, the filtering for a θ can be applied parallelly before the capture of projection at another θ is completed.

Equation (18) indicates the inverse Fourier transformation of the product of $G_\theta(\xi)$, which is the Fourier transform of the Radon transform $g(s, \theta)$, and the filter function $|\xi|$. If we employ $FT^{-1}[|\xi|]$, which is the filter function in the real domain, we get

$$\hat{g}(s, \theta) = g(s, \theta) * FT^{-1}[|\xi|] \tag{20}$$

This is a simple convolution and the Fourier transformation is not required. This method is called *convolution back-projection method.*

Realization of the filter function

We should discuss how to realize the filter that multiplies $|\xi|$ in the frequency domain. Since the gain of this filter is proportional to $|\xi|$, the gain is infinite at the infinite frequency, as shown in Fig. 2(a). Such filter cannot be practically realized.

To solve this problem, the filter should be modified to be defined on a finite support in the frequency domain. Since the filter response at frequencies higher than the highest spatial frequency of the projection, denoted $\xi_{\max}$, is meaningless, the frequency components higher than ξ max can be truncated. We consider truncating the function $|\xi|$ at $\xi_{\max}$, as shown in Fig. 2(b). This is called *Ramachandran-Lakshminarayanan* (or *Ram-Lak*) filter, and its filter function is

$$H(\xi) = |\xi| \mathrm{rect}\left(\frac{\xi}{2\xi_{\max}}\right). \tag{21}$$

Since the Ram-Lak filter emphasizes higher frequencies, it often emphasizes noises in images. Various modifications of this filter for noise reduction by suppressing the gain in high frequencies have been proposed. One typical example is *Shepp-Logan* filter, whose filter function is

$$H(\xi) = |\xi| \mathrm{sinc}\left(\frac{\xi}{2\xi_{\max}}\right) \mathrm{rect}\left(\frac{\xi}{2\xi_{\max}}\right), \tag{22}$$

as shown in Fig. 2(c). This filter is a modified version of the Ram-Lak filter by multiplying the sinc function. Since this multiplication is equivalent to the convolution with the rect function in the real domain, this modification is equivalent to the average filtering in the real domain.