2007 Autumn semester Pattern Information Processing

Topic 1. Sampling and digital processing of images

Session 4. (3) Discrete Fourier transformation

Discrete Fourier transformation

It is often easier to treat continuous functions than discrete ones in mathematics; however, the digital computer can handle discrete functions only. In this section, we consider the Fourier transformation of sampled discrete functions.

As defined in the previous session, the Fourier transform of the sampled function $f_T(x)$ is as follows:

$$\begin{aligned} & FT[f_T(x)](\nu) \\ = \; & FT[f(x)\mathrm{comb}_T(x)](\nu) \\ = \; & \int_{-\infty}^{\infty} f(x)\mathrm{comb}_T(x)\exp(-i2\pi\nu x)dx. \end{aligned} \quad (1)$$

By the definition of comb function by Eq. (1) in the previous session, we get from Eq. (1) that

$$\begin{aligned} & FT[f_T(x)](\nu) \\ = \; & \int_{-\infty}^{\infty} f(x)\sum_{n=-\infty}^{\infty}\delta(x-nT)\exp(-i2\pi\nu x)dx \\ = \; & \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\{f(x)\exp(-i2\pi\nu x)\}\delta(x-nT)dx. \end{aligned} \quad (2)$$

Similarly to Eq. (9) in the previous session, which was a property of the convolution by the delta function, the the integral in Eq. (2) is rewritten as follows:

$$\begin{aligned} & \int_{-\infty}^{\infty}\{f(x)\exp(-i2\pi\nu x)\}\delta(x-nT)dx \\ = \; & \int_{-\infty}^{\infty}\{f(nT)\exp(-i2\pi\nu x)\}\delta(nT-nT)dx \\ = \; & f(nT)\exp(-i2\pi\nu nT). \end{aligned} \quad (3)$$

Thus we get from Eq. (2) that

$$FT[f_T(x)](\nu) = \sum_{n=-\infty}^{\infty} f(nT)\exp(-i2\pi\nu nT). \quad (4)$$

Since $f_T(x)$ is sampled from $f(x)$, $f(nT) = f_T(nT)$. Thus we get from Eq. (13) that

$$FT[f_T(x)](\nu) = \sum_{n=-\infty}^{\infty} f_T(nT)\exp(-i2\pi\nu nT). \quad (5)$$

$f_T(x)$ has been regarded as the sampled version of $f(x)$ so far; Now we consider the situation that only $f_T(x)$ is given and the original $f(x)$ is unknown. In this case $f_T(x)$ is simply a sequence of numbers. We also rename the sampling period T([meters]) to 1([unit length] or [step]), since one step of the sequence corresponds to length T of $f_T(x)$. Then we get the sequence $u(n)$ as follows:

$$u(n) = f_T(nT). \quad (6)$$

Consequently, one step of the sequence $u(n)$ corresponds to length T([meters]) in $f_T(x)$. Then rewriting Eq. (5), we define $\tilde{U}(\nu)$ as follows:

$$\tilde{U}(\nu) = \sum_{-\infty}^{\infty} u(n)\exp(-i2\pi\nu n). \quad (7)$$

$\tilde{U}(\nu)$ is regarded as *the Fourier transform of sequence u(n)*[1]. As explained above, the Fourier transform of a sampled function is periodic in the frequency domain. Since the sampling period is 1([unit length]), the period of the Fourier transform is 1(1/[unit length]). We consider only one period in the frequency domain in the followings.

A continuous function is obtained in the frequency domain by the Fourier transformation of a sequence, although the original function is discrete in the real domain. It is inconvenient for digital computation. Thus we take N samples from $\tilde{U}(\nu)$, i. e. the sampling rate is $1/N$ (1 / [unit length]). We get from this sampling the following:

$$\tilde{U}(\frac{k}{N}) = \sum_{-\infty}^{\infty} u(n)\exp(-i2\pi\frac{k}{N}n) \;\; (k = 0, 1, \ldots, N-1). \quad (8)$$

What happens in the real domain by this frequency domain sampling? The answer is obtained by the reverse operation of the sampling theorem. The operation to obtain the real domain function $f(x)$ from

[1]This operation is also refered as *the discrete time Fourrier transformation (DTFT) of function $u(n)$ of discrete time n*, since the axis of real domain is often the time for one dimensional functions.

the frequency domain function $F(\nu)$ is called *inverse Fourier transformation.* As explained in the previous session, since the component of frequency ν in the frequency domain corresponds to the coefficient of $\exp(i2\pi\nu x)$, the inverse Fourier transformation is defined as follows [2]:

$$FT^{-1}[F(\nu)](x) = \int_{-\infty}^{\infty} F(\nu)\exp(i2\pi\nu x)d\nu = f(x) \tag{9}$$

This is very similar to the original Fourier transformation. Thus the discussion about the sampling theorem also holds for the inverse Fourier transformation, i. e. sampling in the frequency domain by interval $1/N$ yields the periodic duplications of the original function at period N in the real domain. Thus the sampling of the Fourier transform of a sequence at period $1/N$ in the frequency domain corresponds to the Fourier transform of the N-point sequence extended to the periodic function by the infinite duplications of the sequence in the real domain. If the original sequence contains exactly N numbers, the discrete Fourier transform of the original sequence is properly obtained by the sampling at period $1/N$ in the frequency domain, in the sense of obtaining the Fourier transform of the periodic extension of the sequence.

Now we assume that $u(n)$ is a finite sequence of N numbers, i. e. $u(n) = 0$ except $n = 0, 1, \ldots, N?1$. Then we get from Eq. (8) that

$$\tilde{U}(\frac{k}{N}) = \sum_{n=0}^{N-1} u(n)\exp(-i2\pi\frac{k}{N}n) \ \ (k = 0, 1, \ldots, N-1). \tag{10}$$

Renaming the left side to $U(k)$, we get

$$U(k) = \sum_{n=0}^{N-1} u(n)\exp(-i2\pi\frac{k}{N}n) \ \ (k = 0, 1, \ldots, N-1). \tag{11}$$

This operation is called *discrete Fourier transformation (DFT)*. The Fourier transformation perfomed by digital computers is always the discrete Fourier transformation.

As an example, we assume a one-dimensional signal $u(n)$ of $N = 256$ points sampled at period $T = 1$([milimeter]). Now we consider the period equivalent to one step of k in the frequency domain, if we get $U(k)$ from $u(n)$ by the discrete Fourier transformation. The Fourier transformation of a signal sampled at period T([meters]) yields the repetition of the Fourier transform of the original signal by period $1/T$([1/meter]) in the frequency domain. Since one step of the discrete Fourier transform is equal to one of N divisions of the period of $1/T$([1/meter]), one step corresponds to 1/ NT([1/meter]). Thus one step corresponds to $1/256$([1/ milimeter]) in this example, since $T = 1$([milimeter]) and $N = 256$([steps]). Figure 1 illustrates this relationship.

Note that $U^*(N?k) = U(k)$ for the discrete Fourier transformation of a sequence of N real numbers. It is proved as follows: Let $U^*(k)$ be the complex conjugate of $U(k)$. If $u(n)$ is a sequence of N real numbers, we get

$$\begin{aligned} U^*(N-k) &= \sum_{n=0}^{N-1} u^*(n)\exp(i2\pi\frac{N-k}{N}n) \\ &= \sum_{n=0}^{N-1} u(n)\exp(i2\pi n)\exp(i2\pi\frac{-k}{N}n). \end{aligned} \tag{12}$$

Since $\exp(i2\pi n) = 1$ for integer n, we get

$$\begin{aligned} U^*(N-k) &= \sum_{n=0}^{N-1} u(n)\exp(i2\pi\frac{-k}{N}n) \\ &= U(k). \end{aligned} \tag{13}$$

This relationship means that the highest frequency that can be expressed by the discrete Fourier transformation of N points is $N/2$, and only $N/2+1$ numbers from $U(0)$ to $U(N/2)$ are unique. It is also found that the maximum frequency corresponds to $k = N/2$, i. e. the middle of the sequence, and zero frequency corresponds to $k = 0$, i. e. the end of the sequence.

[2]The right side of this equation is divided by N for normalization in some textbooks. This is explained later in the section of the unitary transformation.

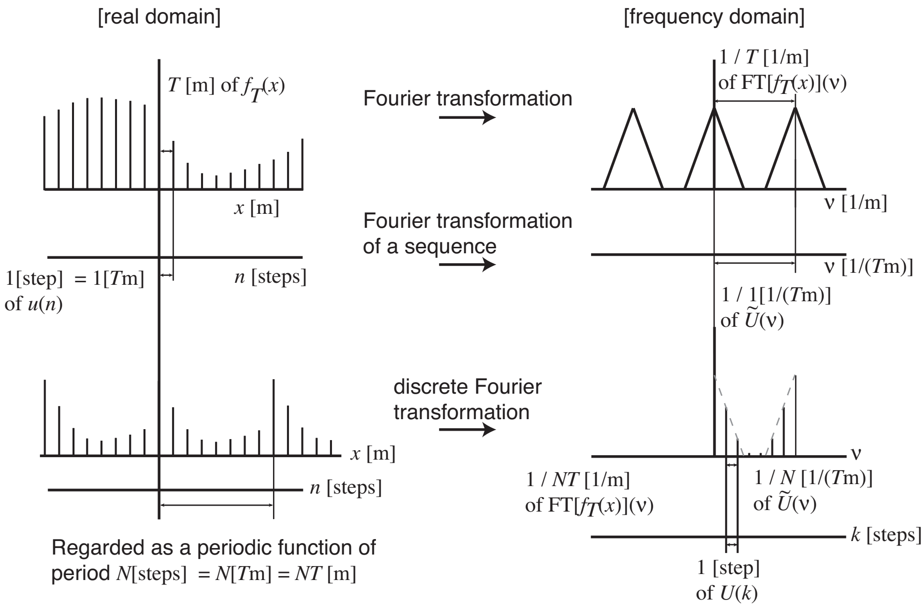

Fig. 1: Discrete Fourier transformation.