Session 2. (1) Spatial frequency and Fourier series expansion

Organization of digital images is explained in this topic. Although an image is naturally a continuous distribution of brightness, it has to be converted into a discrete set of integers for computer processing. The conversion into discrete pixels is called *sampling,* and the conversion into integers is called *quantization.* The sampling period is quite important issue, and it is evaluated by the concept of *spatial frequency.* The concept of spatial frequency and Fourier transformation are explained in this topic.

Diffraction of light and imaging

We start from the optical phenomenon of imaging. The imaging is defined as a collection of diverged light spread from a point of object into one point by a lens. This phenomenon can be observed from the following point of view: The light has the property of *diffraction.* The diffraction of wave is a phenomenon that the wave reaches beyond an opaque object obstracting its path. For example, even if the progress of wave on a water surface is obstructed by a board, it reaches beyond the board. Since the light is a kind of electromagnetic wave, the light has this property. The radio wave reaches beyond obstructing objects from the broadcasting station by diffraction.

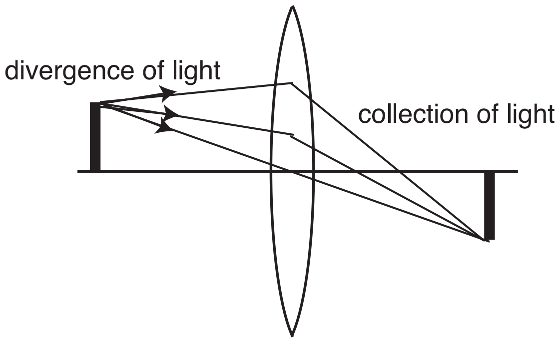

Fig. 1: Imaging.

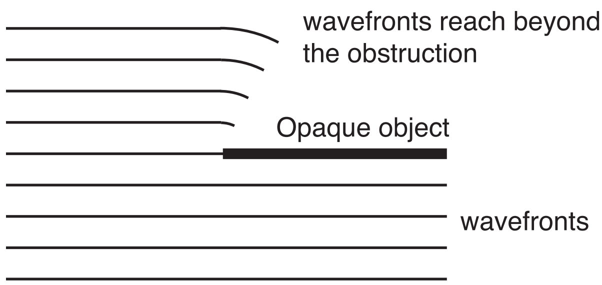

Fig. 2: Diffraction.

If the light passes through a *diffraction grating,* which is an object whose transparence is changing periodically or where transparent and opaque bands are aligned one after another, the light that passes through a transparent band interferes with the light that passes the other transparent bands. Since the light waves passing through adjacent bands emphasize each other along the direction such that the phase shift of the light waves is exactly the same as the wavelength, a bright light, called the *diffracted light of first order,* is obtained in this direction. The smaller the period of bands, the larger the angle between the diffracted light and the light passing directly through the grating (called the *zeroth order light*). If the grating contains pure opaque and pure transparent bands only, diffracted lights along several directions are obtained. However, if the transparence of grating is sinusoidally distributed, diffracted lights of zeroth and first orders only are obtained.

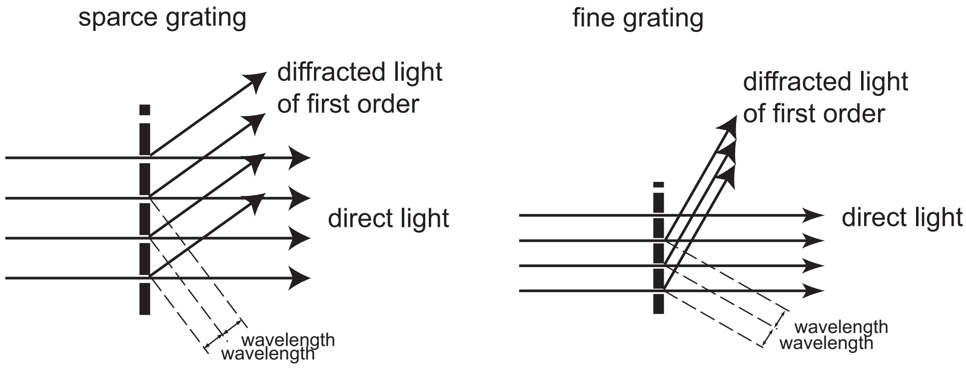

Fig. 3: Diffraction grating and diffracted light of first order.

Suppose that a figure on a transparent film is illuminated by a plain light wave[1]. Suppose also the figure is organized by a superposition of many sinusoidal wave of transparence, i.e. superposition of many diffraction gratings. Each grating diffracts the incident light and produces the diffracted light. The smaller the period of grating is, the larger the angle of diffraction is.

If these diffracted lights pass through the imaging lens, each diffracted light is refracted and interferes with the zeroth order light at the image plane. This interference produces a stripe, or *interference* fringe, on the image plane. The larger the angle of diffracted light is, the smaller the period of stripe is. The distribution of transparence on the film is reconstructed on the image plane as the superposition of these stripes.

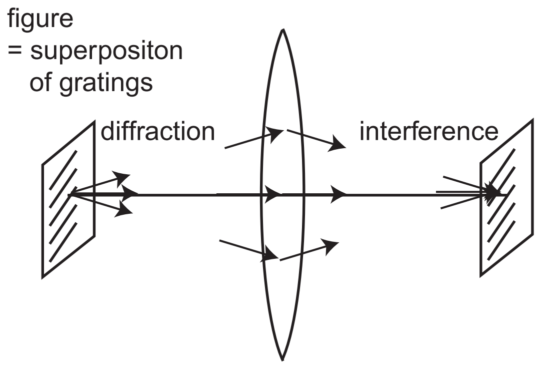

Fig. 4: Imaging by diffraction and interference.

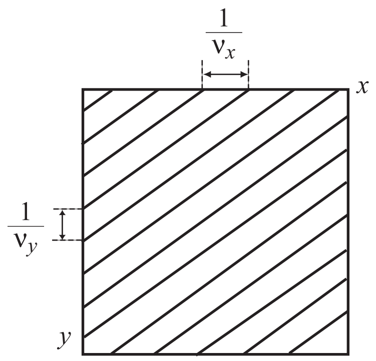

Fig. 5: Spatial frequency.

Spatial frequency

Understanding the process of imaging, the figure on the film is regarded as a superposition of sinusoidal waves of transparence. The number of repetition of the sinusoidal wave per unit length is called

[1]The following explanation is in case of coherent illumination such as lasers. It is more complicated in the case of ordinary incoherent illumination.

spatial frequency. The unit of spatial frequency is cycle/m in the MKSA unit system. Note that the wave is "a wave on the plane." Since the wave on the plane has its direction, the spatial frequency is described as a set of ν_x and ν_y which are frequencies along x-direction and y-direction, respectively. The figure on a film is decomposed into a set of waves of various spatial frequencies. The amplitude of wave at a specific spatial frequency is called *component* corresponding to the spatial frequency.

Fourier series

It is shown in the previous section that a figure on a film can be decomposed into spacial frequency components. *Fourier transformation* is the operation to calculate the components. The principle of Fourier transformation will be explained in the following way: The transparence distribution of the figure on the film can be regarded as a mathematical function. Here we assume onedimensional functions for simplicity. Let us assume that $f(x)$ is a periodic function of period L. We also assume that $f(x)$ is expressed as a superposition of sinusoidal waves of various frequencies. In this case, periods of all the superposed waves should be also L; Otherwise the superposed waves displace each other for large x. Only the waves of periods $L/2, L/3, L/4, \ldots, L/n, \ldots$, where n is an integer, have the period L. It indicates that the number of the superposed waves can be infinite but countable, and the periods $L/2, L/3, L/4, \ldots, L/n, \ldots$ are discrete. Thus it is possible to express this superposition by a sum of the infinite terms, i. e. a *series.*

The sinusoidal wave of period L/n is expressed using the exponential function as $\exp(i2\pi\frac{n}{L}x)$, where i indicates the imaginary unit. The multiplication of 2π expresses the frequency in radian per unit length. This value, $2\pi\frac{n}{L}$, is called *angular frequency.*

The function $f(x)$ is expressed using the above exponential function as the following series:

$$f(x) = \sum_{n=-\infty}^{\infty} a_n \exp\left(i2\pi\frac{n}{L}x\right). \tag{1}$$

The above exponential function has a property that the integral of the multiplication of the functions with an identical period in the range of $[-L/2, L/2]$ yields nonzero value and that of the functions with different periods yields zero. This is because the above integral is expressed as

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} \exp\left(i2\pi\frac{m}{L}x\right)\exp\left(-i2\pi\frac{n}{L}x\right)dx = \int_{-\frac{L}{2}}^{\frac{L}{2}} \exp\left(i2\pi\frac{m-n}{L}x\right)dx, \tag{2}$$

where m, n are integers. If $m \neq n$, we get [2]

$$\begin{aligned} &\frac{1}{i2\pi\frac{m-n}{L}}\left[\exp\left(i2\pi\frac{m-n}{L}x\right)\right]_{-\frac{L}{2}}^{\frac{L}{2}} \\ &= \frac{1}{i2\pi\frac{m-n}{L}}\left\{\exp\left(i2\pi\frac{m-n}{L}\frac{L}{2}\right) - \exp\left(-i2\pi\frac{m-n}{L}\frac{L}{2}\right)\right\} \\ &= \frac{L}{m-n}\sin\pi(m-n)L = 0, \end{aligned} \tag{3}$$

and if $m = n$, we get

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} \exp(0)dx = [x]_{-\frac{L}{2}}^{\frac{L}{2}} = L. \tag{4}$$

This integral is called the *inner product* of two waves, and the group of functions having the above property is called the *orthogonal function system.* This is explained again in the section for image compression in Topic. 2.

Now we calculate here the following operation.

$$\frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\exp\left(-i2\pi\frac{k}{L}x\right)dx, \tag{5}$$

where k is an integer. Since $f(x)$ is expressed by the series in Eq. (1), it follows from the above integral that

$$\frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \sum_{n=-\infty}^{\infty} a_n \exp\left(i2\pi\frac{n}{L}x\right)\exp\left(-i2\pi\frac{k}{L}x\right)dx. \tag{6}$$

Because of the orthogonality shown above, the integrals except $n = k$ yields zero, and it follows from Eq. (6) that

$$\frac{1}{L}\cdot La_k = a_k. \tag{7}$$

It indicates that the coefficient of each wave is obtained by Eq. (6) when $f(x)$ is expressed by the series in Eq. (1). The series in Eq. (1) is called *Fourier series expansion* and the coefficient obtained in Eq. (6) is called *Fourier coefficient* of the wave of period L/k.

[2] The minus sign on $\exp\left(-i2\pi\frac{n}{L}x\right)$ indicates the complex conjugate.

Appendix. Cosine wave and exponential function

In the explanation of Fourier series expansion, a cosine wave is expressed by an exponential function. Why is not a cosine function itself used? This is just because the calculation of exponential functions is simpler than that of trigonometric functions. The relationship between these two functions and the meaning of negative frequencies are explained in this appendix.

From Euler's equation,

$$\exp(i\omega) = \cos\omega + i\sin\omega, \tag{A1}$$

we get the following relationship between the trigonometric functions and the exponential function:

$$\begin{aligned} \cos\omega &= \frac{\exp(i\omega) + \exp(-i\omega)}{2} \\ \sin\omega &= \frac{\exp(i\omega) - \exp(-i\omega)}{2i}. \end{aligned} \tag{A2}$$

From Eq. (A1), a cosine wave $a_1 \cos 2\pi\nu_1 x$ in the real domain is expressed using exponential functions as follows:

$$a_1 \cos 2\pi\nu_1 x = \frac{a_1}{2}\exp(i2\pi\nu_1 x) + \frac{a_1}{2}\exp(i2\pi(-\nu_1)x), \tag{A3}$$

and in its Fourier series expantion we get two peaks of height $a_1/2$ at frequencies ν_1 and $-\nu_1$. This shows that *one cosine wave corresponds to a combination of positive and negative frequencies.*

On the other hand, considering the wave of the same amplitude and frequency with phase shift θ, we get

$$\begin{aligned} & a_1 \cos(2\pi\nu_1 x + \theta) \\ = & \frac{a_1}{2}\exp(i(2\pi\nu_1 x + \theta)) \\ & + \frac{a_1}{2}\exp(-i(2\pi\nu_1 x + \theta)) \\ = & \frac{a_1}{2}\exp(i2\pi\nu_1)\exp(i\theta) \\ & + \frac{a_1}{2}\exp(i2\pi(-\nu_1)x)\exp(-i\theta) \end{aligned} \tag{A4}$$

In this case, the amplitudes of peaks at ν_1 and $-\nu_1$ are $\frac{a_1}{2}\exp(i\theta)$ and $\frac{a_1}{2}\exp(-i\theta)$, respectively. In the case of Eq. (A4), considering the axes of complex amplitude and phase, we get that the amplitude is the same as Eq. (A3), however, the additional peaks of height θ along the axis of complex phase appear at ν_1 and $-\nu_1$. This example shows that a phase shift of sinusoidal waves appears as a difference in complex phase.

Reference

J.W.Goodman, *Introduction to Fourier Optics,* McGraw-Hill.

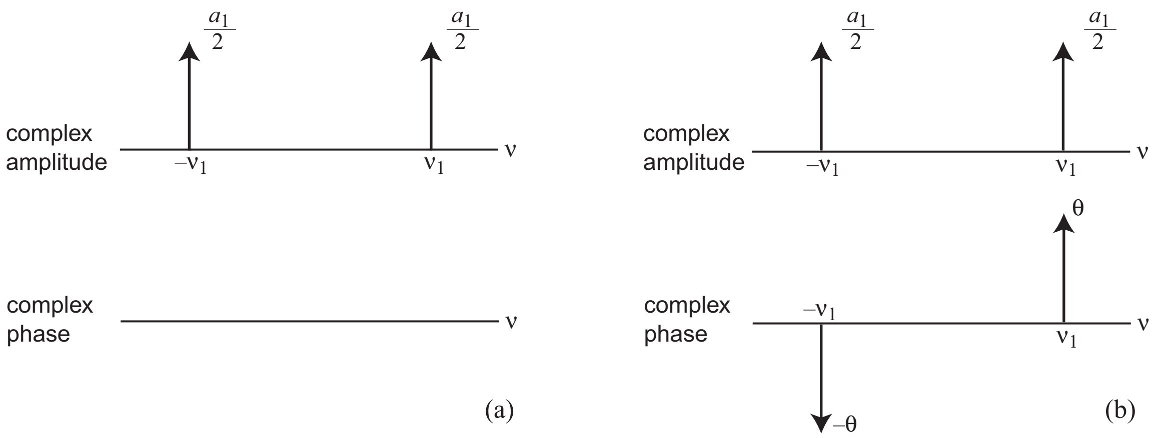

Fig. 1: Expressions of phases. (a) $a_1 \cos 2\pi\nu_1 x$. (b) $a_1 \cos(2\pi\nu_1 x + \theta)$.