2009 Autumn semester Pattern Information Processing

Topic 3. Mathematical morphology

Session 8. (1) Opening and set operations on images

Mathematical morphology treats an effect on an image as an effect on the shape and size of objects contained in the image. Mathematical morphology is a mathematical system to handle such effects quantitatively based on set operations. The word stem "morpho-" originates in a Greek word meaning "shape," and it appears in the word "morphing," which is a technique of modifying an image into another image smoothly.

The founders of mathematical morphology, G. Mathéron and J. Serra, were researchers of l'École Nationale Supérieure des Mines de Paris in France, and had an idea of mathematical morphology as a method of evaluating geometrical characteristics of minerals in ores. Mathéron is also the founder of the randam closed set theory, which is a fundamental theory of treating random shapes, and kriging, which is a statistical method of estimating a spatial distribution of mineral deposits from trial diggings.

Mathematical morphology has relationships to these theories and has been developed as a theoretical framework of treating spatial shapes and sizes of objects. The International Symposium on Mathematical Morphology (ISMM), which is the topical international symposium focusing on mathematical morphology only, has been organized almost every two years, and its seventh symposium was held in April 2005 in Paris as a cerebration of 40 years anniversary of mathematical morphology.

This topic contains the followings: (1) opening, which is the fundamental concept of mathematical morphology, and set operations of image objects, (2) granulometry, which treats "size" of image objects quantitatively, and (3) filter theorem, which unifies image shape transformations under the mathematical morphology, and the mathematical morphology for color images based on the concept of lattice.

Opening

The fundamental operation of mathematical morphology is "opening," which discriminates and extracts object shapes with respect to the size of objects. We explain opening on binary images at first, and basic operations to describe opening.

In the context of mathematical morphology, an object in a binary image is regarded as a set of vectors corresponding to the points composing the object. In the case of usual digital images, a binary image is expressed as a set of white pixels or pixels of value one. Another image set expressing an effect to the above image set is considered, and called *structuring element.* The structuring element corresponds to the window of an image processing filter, and is considered to be much smaller than the target image to be processed.

Let the target image set be X, and the structuring element be B. *Opening* of X by B has a property as follows:

$$X_B = \{B_z \,|\, B_z \subseteq X, z \in \mathbb{Z}^2\}, \tag{1}$$

where B_z indicates the *translation* of B by z, defined as follows:

$$B_z = \{b + z \,|\, b \in B\}. \tag{2}$$

This property indicates that the opening of X with respect to B indicates the locus of B itself sweeping all the interior of X, and removes smaller white regions than the structuring element, as illustrated in Fig. 1. Since opening eliminates smaller structures and smaller bright peaks than the structuring element, it has a quantitative smoothing ability. It also indicates that the difference between X and X_B is the structure which is smaller than B in X, since it is the residue of X which is too small to be drawn by the sweeping B.

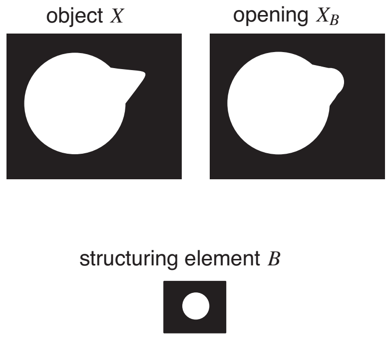

Fig. 1: Effect of opening.

Opening decomposed to erosion and dilation

Although the property of opening in Eq. (1) is intuitively understandable, this is not a pixelwise operation. Thus opening is defined by a composition of simpler pixelwise operations. In order to define opening, *Minkowski set subtraction* and *addition* are defined as the fundamental operations of mathematical morphology.

$$X \ominus B = \bigcap_{b \in B} X_b, \tag{3}$$

$$X \oplus B = \bigcup_{b \in B} X_b. \tag{4}$$

Minkowski set subtraction has the following property: It follows from $x \in X_b$ that $x - b \in X$. Thus the definition of Minkowski set subtraction in Eq. (3) can be rewritten to the following pixelwise operation:

$$X \ominus B = \{x | x - b \in X, b \in B\}. \tag{5}$$

The *reflection* of B, denoted $\check{B}$, is defined as follows:

$$\check{B} = \{-b | b \in B\} \tag{6}$$

Since it follows from the above two equations that $X \ominus B = \{x | (-b) + x \in X, (-b) \in \check{B}\}$, Minkowski set subtraction is expressed as follows:

$$X \ominus B = \{x | \check{B}_x \subseteq X\}. \tag{7}$$

Since we get from the definition of reflection in Eq. (6) that $\check{B}_x = \{-b + x | b \in B\}$, it follows that $\check{B}_x = \{x - b | b \in B\}$. We get the relationship in Eq. (7) by substituting it into Eq. (5). This relationship indicates that $X \ominus B$ is the locus of the origin of $\check{B}$ when $\check{B}$ sweeps all the interior of X.

For Minkowski set addition, it follows that

$$\bigcup_{b \in B} X_b = \{x + b | x \in X, b \in B\}. \tag{8}$$

Thus we get

$$X \oplus B = \{b + x | b \in B, x \in X\} = \bigcup_{x \in X} B_x. \tag{9}$$

It indicates that $X \oplus B$ is composed by pasting a copy of B at every point within X.

Using the above operations, *erosion* and *dilation* of X with respect to B are defined as $X \ominus \check{B}$ and $X \oplus \check{B}$, respectively.

We get from Eq. (7) that

$$X \ominus \check{B} = \{x | B_x \subseteq X\}. \tag{10}$$

It indicates that $X \ominus \check{B}$ is the locus of the origin of B when B sweeps all the interior of X.

The opening X_B is then decomposed to the above fundamental operations as follows:

$$X_B = (X \ominus \check{B}) \oplus B. \tag{11}$$

The equivalence of the above expression and Eq. (1) is proved as follows: It is sufficient to prove $x \in X \ominus \check{B} \Leftrightarrow B_x \subseteq X$, since X_B is defined here as $(X \ominus \check{B}) \oplus B$ and it is equivalent to $\bigcup_{z \in X \ominus \check{B}} B_z$, and stated in Eq. (1) as $X_B = \{B_z | B_z \subseteq X\}$.

(1) $[x \in X \ominus \check{B} \Rightarrow B_x \subseteq X]$:
$b \in B \Leftrightarrow b + x \in B_x, \ -b \in \check{B}$.
$x \in X \ominus \check{B} \Rightarrow x - (-b) = b + x \in X$ for all $(-b) \in \check{B}$.
Thus $x \in X \ominus \check{B} \Rightarrow B_x \subseteq X$.

(2) $[B_x \subseteq X \Rightarrow x \in X \ominus \check{B}]$:
$B_x \subseteq X \Rightarrow b + x \in X$ for all $b \in B$.
$\Rightarrow$ For all $(-b) \in \check{B}, \ x - (-b) \in X$.
$\Rightarrow x \in X \ominus \check{B}$.

The above definition of opening is illustrated in Fig. 2. A black dot indicates a pixel composing an image object in this figure. As shown in the above, the erosion of X by B is the locus of the origin of B when B sweeps all the inside of X. Thus the erosion in the first step of opening produces every point where a copy of B included in X can be located. The Minkowski addition in the second step locates a copy of B at every point within $X \ominus \check{B}$. Thus the opening of X with respect to B indicates the locus of B itself sweeping all the interior of X, as described at the beginning of this section. In other words, the opening removes regions of X which are too small to include a copy of B and preserves the others. The figure also indicates the difference between the opening and the "erosion followed by dilation."

The counterpart of opening is called *closing,* defined as follows:

$$X^B = (X \oplus \check{B}) \ominus B. \tag{12}$$

The closing of X with respect to B is equivalent to the opening of the background, and removes smaller spots than the structuring element within image objects. This is because the following relationship between opening and closing holds:

$$[X^B]^c = (X^c)_B, \tag{13}$$

where X^c indicate the *complement* of X and defined as $\{x | x \notin X\}$. The relationship of Eq. (13) is called *duality* of opening and closing[1].

The above duality is proved from the duality of Minkowski addition and subtraction, i. e. $X \oplus B = (X^c \ominus B)^c$. This property is proved as follows:

Proof:

[1]There is another notation system which denotes opening as $X \circ B$ and closing as $X \bullet B$.

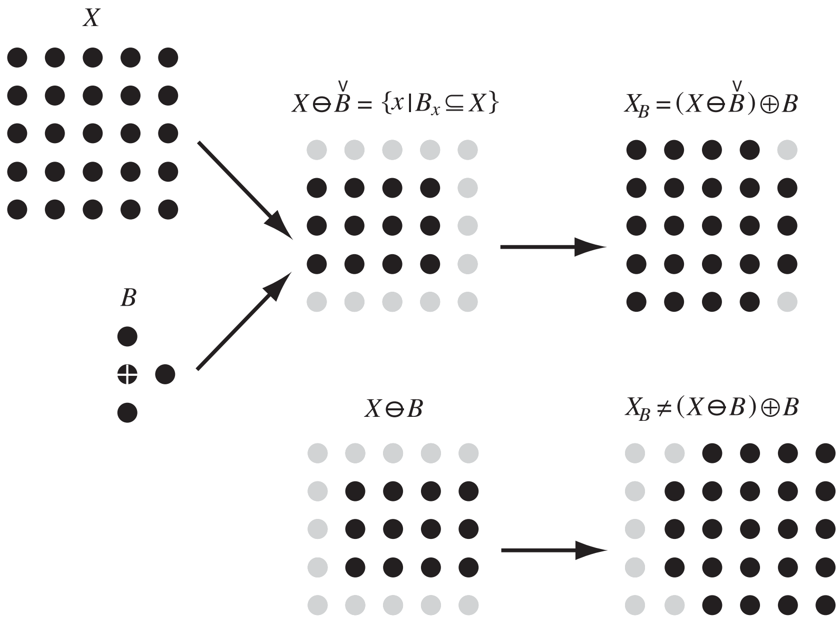

Fig. 2: opening composed of fundamental operations.

$$\begin{aligned}
Proof:\ z \in (X^c \ominus B)^c &\Leftrightarrow z \notin X^c \ominus B. \\
z \notin X^c \ominus B &\Leftrightarrow z \in (X^c)_b \text{ for all } b \in B. \\
\text{Thus } z \in (X^c)_b &\Leftrightarrow \text{There exists } b \in B \text{ such that } z \notin (X^c)_b. \\
&\Leftrightarrow \text{There exists } b \in B \text{ such that } z \in X_b. \\
&\Leftrightarrow z \in X \oplus B.
\end{aligned}$$

Figure 3 summarizes the illustration of the effects of basic morphological operations[2].

Extension to gray scale images

In the case of gray scale image, an image object is defined by the *umbra* set. If the pixel value distribution of an image object is denoted as $f(x)$, where $x \in \mathbb{Z}^2$ is a pixel position, its umbra $U[f(x)]$ is defined as follows:

$$U[f(x)] = \{(x,t) \in \mathbb{Z}^3 \mid -\infty < t \leq f(x)\}. \tag{14}$$

Consequently, when we assume a "solid" whose support is the same as a gray scale image object and whose height at each pixel position is the same as the pixel value at this position, the umbra is equivalent to this solid and the whole volume below this solid within the support, as illustrated in Fig. 4.

A gray scale structuring element is also defined in the same manner. Let $f(x)$ be the gray scale pixel value at pixel position x and $g(y)$ be that of the structuring element. Erosion of f by g is defined for the umbrae similarly to the binary case, and reduced to the following operation.

$$\{f \ominus \breve{g}\}(x) = \inf_{b \in w(g)} \{f(x+b) - g(b)\}. \tag{15}$$

[2] There is another definition of morphological operations which denotes the erosion in the text as $X \ominus B$ and call the Minkowski set addition in the text "dilation." The erosion and dilation are not dual in this definition, however, it has an advantage that opening is simply denoted as $(X \ominus B) \oplus B$, i. e. "erosion followed by dilation."

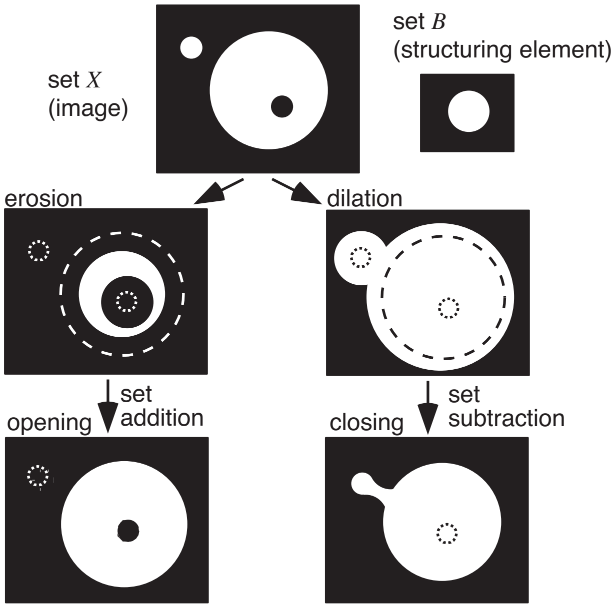

Fig. 3: Effects of erosion, dilation, opening, and closing

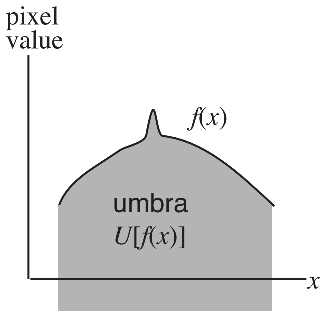

Fig. 4: Umbra. The spatial axis x is illustrated one-dimensional for simplicity.

Dilation is also reduced to

$$\{f \oplus \check{g}\}(x) = \sup_{b \in w(g)} \{f(x+b) + g(b)\}. \tag{16}$$

where $w(g)$ is the support of g.

These equations indicate that the logical AND and OR operations in the definition for binary images are replaced with the infimum and supremum operations (equivalent to minimum and maximum in the case of digital images), respectively.

Expanding the idea, mathematical morphological operations can be defined for sets where the infimum and supremum among the elements are defined in some sense. For example, morphological operations for color images cannot be defined straightforwardly from the above classical definitions, since a color pixel value is defined by a vector and the infimum and supremum are not trivially defined. The operations can be defined if the infimum and supremum among colors are defined. Such set is called *lattice,* and mathematical morphology is generally defined as operations on a lattice. This will be explained in the third session of this topic.