2009 Autumn semester Pattern Information Processing
Topic 3. Mathematical morphology

Session 9. (2) Granulometry and skeleton

In the second session of this topic, we explain *granulometry* and *size distribution* for treating size of objects quantitatively, and the concept of *skeleton*. Granulometry expresses how much portion of each size is contained in an object. This is the most essential part of mathematical morphology, since the origin of mathematical morphology comes from a method of mineral particle analysis developed by G. Matheron and J. Serra with the École de Mine du Paris. The theoretical development of mathematical morphology has begun from the requirement of quantitative estimation of particle shape and size for mineral analysis. Matheron also makes a lot of contribution to the area of *spatial statistics* including *kriging*, which is a method of estimating the resource distribution from geographical measurements at several points, and *random set theory,* which is a framework of probabilistic geometry.

Defintion of size

The "size" in the sense of the mathematical morphology is defined as the magnification ratio between a basic-shape object (expressed as a set) and its homothetic magnification. For example, if a circle whose diameter is 1 cm is defined as "the circle of size 1," the circle of size 2 is the circle whose diameter is 2 cm.

More generally, if we assume a continuous object set B, "r-times magnified B" is defined as follows:

$$rB = \{rb | b \in B\}, \tag{1}$$

If B is defined as an object of size 1, the size of rB is r.

This definition is, however, not applicable to discrete sets. If $2B$ is defined by Eq. (1) for a discrete set B, $2B$ will contain unnecessary spaces, as shown in the top center of Fig. 1. To avoid this, in case of discrete sets, rB is defined by the Minkowski set addition, as follows:

$$rB = B \oplus B \oplus \cdots \oplus B \quad ((r-1) \text{ times}). \tag{2}$$

Figure 1 shows $2B$ by this definition.

Note that B in Fig. 1(b) is a rhombus as shown by $2B$, although it resembles a cross as B in Fig. 1.

Granulometry and size distribution

It was explained in the previous session that "the opening of image X by structuring element B" means "removing smaller portions than B from X." It indicates that the opening works as a filter to distinguish portions of objects by their "sizes."

Let B be a basic structuring element, and we produce homothetic structuring elements of increasing sizes, $2B, 3B, \ldots$. We then perform opening of X by the sequence of the homothetic structuring elements, and obtain the image sequence $X_B, X_{2B}, X_{3B}, \ldots$. In this sequence, X_B is obtained by removing the regions smaller than B, X_{2B} is obtained by removing the regions smaller than $2B$, X_{3B} is obtained by removing the regions smaller than $3B$, ... The sequence of the structuring elements $B, 2B, \ldots$ should hold that $(n+1)B_{nB} = (n+1)B$. It is called that $(n+1)B$ is *open* with respect to nB, and in this case it holds that $X \supseteq X_B \supseteq X_{2B} \supseteq X_{3B} \supseteq \ldots$. The sequence $B, 2B, \ldots$ is usually composed by a convex structuring element and the homothetic magnification defined in the previous section.

This sequence of openings $X_B, X_{2B}, \ldots$ is called *granulometry*. We then calculate the ratio of the area of X_{rB} to that of the original X at each r. The area of an image is defined by the area occupied by an image object, i. e. the number of pixels composing an image object in the case of discrete images. The function from a size r to the corresponding ratio is monotonically decreasing, and unity when the size is zero. This function is called the size distribution function. The *size distribution function* of size r indicates **the area ratio of the regions whose sizes are greater than or equal to** r.

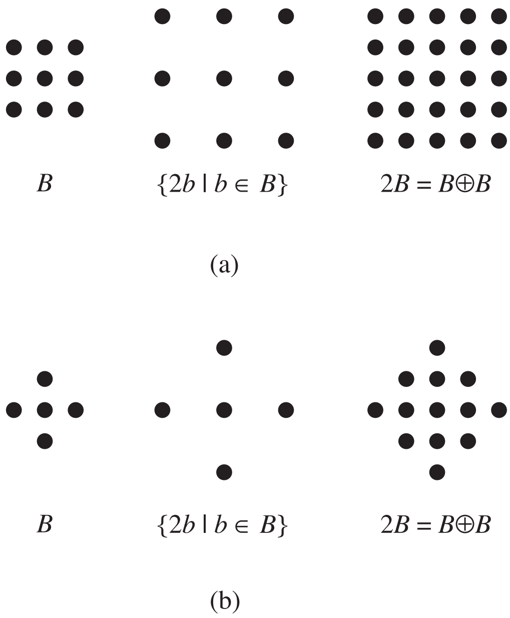

Fig. 1: Definition of size in case of discrete images. (a) square. (b) rhombus.

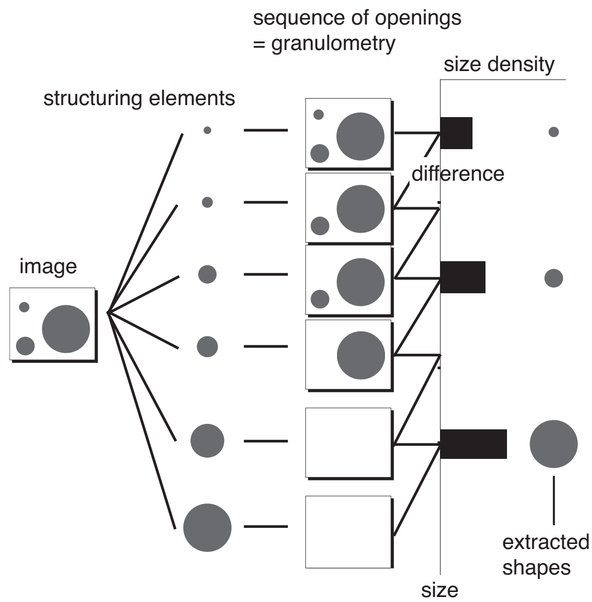

Fig. 2: Granulometry and size distribution.

We consider further the derivative of the size distribution function. In discrete case, it is equivalent to the difference between the areas of adjacent sizes in the granulometry. For example, if we consider the difference between X_{2B} and X_{3B},

The region composing X_{2B} but not composing X_{3B} is

the region not removed by opening by $2B$, but removed by opening by $3B$

i. e. **the region whose size is exactly 2.**

The derivative, i. e. the function from a size to the area ratio corresponding to the size, is called the **size density function.** The above discussion suggests that the size distribution function and the size density function have similar properties to the probability distribution function and the probability density function, respectively.

More formally, the size distribution function of image X by structuring element B is as follows:

$$F_{X,B}(r) = \frac{A(X_{rB})}{A(X)}, \tag{3}$$

where r is a size and $A()$ indicates the area of image objects. The size density function is defined in the case of continuous images as follows:

$$p_{X,B}(r) = \frac{d}{dr}\left(1 - F_{X,B}(r)\right) = -\frac{1}{A(X)}\frac{dA(X_{rB})}{dr}, \tag{4}$$

and in the case of discrete images as follows:

$$\begin{aligned} p_{X,B}(r) &= (1 - F_{X,B}(r+1)) - (1 - F_{X,B}(r)) \\ &= -\frac{1}{A(X)}\left(A(X_{rB}) - A(X_{(r+1)B})\right). \end{aligned} \tag{5}$$

These definitions are for positive sizes. For negative r, they are defined by replacing the openings with the closings and the sizes r with $|r|$. Since opening and closing are dual, the size distribution and density functions for negative r mean the operations for the backgrounds. Figure 2 illustrates the process of granulometry.

Features derived from size density function

An image can be evaluated by the following features derived from its size density function, similarly to the case of probability density function. The followings show the cases of discrete images only, however, the expressions in the case of continuous images are derived by only replacing the summations with integrations. In the following equations N means the maximum size contained in an image X.

$$\textbf{Size mean} \;\; E(X, B) = \sum_{r=0}^{N} r p_{X,B}(r). \qquad (6)$$

This feature indicates the mean size of the image X by the structuring element B. The mean is the first-order moment, and the second-order moment (variance) and higher-order moments can be defined and the size density function is characterized by these moments. They are called *granulometric moments.*

$$\textbf{Size entropy} \;\; H(X, B) = -\sum_{r=0}^{N} \log p_{X,B}(r). \qquad (7)$$

This feature indicates the average roughness of the image X by the structuring element B. If $H(X, B) = 0$, X contains B of only one size, and the roughness is the minimum. If $H(X, B) = log(N + 1)/(N + 1)$, i. e. the maximum, X contains B' s of all sizes equally and the roughness is the maximum.

Skeleton and medial axis transform

Skeleton in the sense of mathematical morphology means shrinking an image object and deriving its medial axis. The morphological skeleton has a characteristic that the image can be reconstructed from its skeleton.

The skeleton of an image object X by a structuring element B, denoted $SK(X, B)$, is defined as follows:

$$\begin{aligned} S_n(X, B) &= (X \ominus n\check{B}) - (X \ominus n\check{B})_B, \\ SK(X, B) &= \bigcup_n S_n(X, B). \end{aligned} \qquad (8)$$

The object cannot be reconstructed from $SK(X, B)$, however, it can be reconstructed from the set of $S_n(X, B)$. The assignment of the value n to each pixel contained in $S_n(X, B)$ is called *medial axis transformation.* The image X is reconstructed from $S_n(X, B)$ as follows:

$$X = \bigcup_n [S_n(X, B) \oplus nB]\,. \qquad (9)$$

Proof:

$$\begin{aligned} &[S_n(X, B) \oplus nB] \\ &= \left[(X \ominus n\check{B}) - (X \ominus n\check{B})_B\right] \oplus nB \\ &= (X \ominus n\check{B}) \oplus nB - (X \ominus n\check{B})_B \oplus nB \\ &= (X \ominus n\check{B}) \oplus nB - (X \ominus n\check{B} \ominus \check{B} \oplus B) \oplus nB \\ &= X \ominus n\check{B} \oplus nB - X \ominus (n + 1)\check{B} \oplus (n + 1)B \\ &= X_{nB} - X_{(n+1)B}. \end{aligned} \qquad (10)$$

From the above, we get

$$\begin{aligned} &\bigcup_n [S_n(X, B) \oplus nB] \\ &= \bigcup_n [X_{nB} - X_{(n+1)B}] \\ &= (X - X_B) \cup (X_B - X_{2B}) \cup (X_{2B} - X_{3B}) \cup \cdots . \end{aligned} \qquad (11)$$

Since $(A - B) \cup (B - C) = A - C$ and $X_{nB} = \emptyset$ for sufficiently large n, the right side of Eq. (11) equals to X.

Intuitively speaking, Eq. (8) has the following meanings: $X \ominus n\check{B}$ means "the locus of the origin of nB, a homothetic magnification of the structuring element B, when the smallest number of nB' s are employed to cover the whole inside of the object X." In this case $(X \ominus n\check{B}) - (X \ominus n\check{B})_B$ means "the origin of nB located at the corner of X, touching the object edge at more than two points," as shown in Fig. 3. Since Eq. (8) performs this operations from smaller n to larger, the skeleton is "the locus of the origin of nB when nB' s are located at the corners, touching the object edge at more than two points, from smaller n to larger," as shown in Fig. 4.

References

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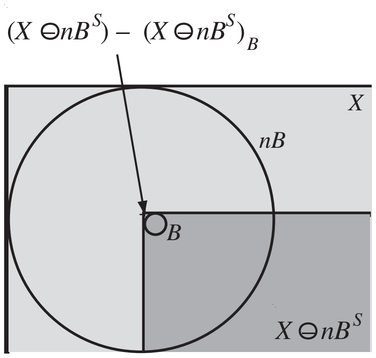

Fig. 3: Schematic illustration of the meaning of Eq. (8).

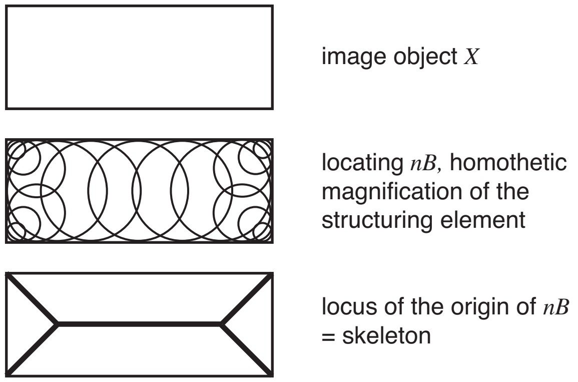

Fig. 4: Derivation of the skeleton.