2010 Autumn semester Pattern Information Processing
Topic 1. Sampling and digital processing of images

Session 3. (2) Fourier transformation and sampling theorem

The Fourier transformation and the sampling theorem will be explained in this session. The Fourier transformation is a general framework to define a conversion of a function, for example an image or a sound signal, to a continous set of frequency components. This is an expansion of the Fourier series expansion for a periodic function to the case of a function with the infinite period.

To convert a continuous distribution of brightness to a digital image, it is necessary to extract the brightness at each point arranged regularly at a period. This operation is called *sampling.* The original continuous distribution of brightness can be reconstructed if the interval is sufficiently small. The sampling theorem gives the maximum period for lossless reconstruction.

Fourier transformation

It was explained in the previous session that a periodic function of period L is decomposed to a Fourier series, which is the summation of countably infinite number of sinusoidal functions. How is the case where $f(x)$ is not periodic? It is considered as the case where the period L tends to the infinity, i. e. $L \to \infty$.

The larger L grows, the smaller the interval between n/L and $(n+1)/L$ becomes. It indicates that the interval of frequencies between two adjacent waves in the Fourier series expression, i. e. $1/L$, becomes smaller.

We define here $\Delta\nu$ as the interval $1/L$. Applying Fourier series expansion, explained in the previous session, and using $\Delta\nu$, the original function $f(x)$ is expressed as follows:

$$f(x) = \sum_{n=-\infty}^{\infty} \left(\Delta\nu \int_{-\frac{L}{2}}^{\frac{L}{2}} f(\tau) \exp(-i2\pi n\Delta\nu\tau)\, d\tau \right) \times \exp(i2\pi n\Delta\nu x)\,. \tag{1}$$

Since the product $n\Delta\nu$ indicates a frequency, it is replaced with ν. When $L \to \infty$, we get $\Delta\nu \to 0$, and the summation including $\Delta\nu$ is regarded as the integral including $d\nu$. We get from the above that

$$f(x) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(\tau) \exp(-i2\pi\nu\tau)\, d\tau \right) \exp(i2\pi\nu x)\, d\nu. \tag{2}$$

The above equation is separated to the following two equations:

$$F(\nu) = \int_{-\infty}^{\infty} f(x) \exp(-i2\pi\nu x)\, dx, \tag{3}$$

$$f(x) = \int_{-\infty}^{\infty} F(\nu) \exp(i2\pi x\nu)\, d\nu. \tag{4}$$

Equation (3) is called *Fourier transformation,* and Eq. (4) is called *inverse Fourier transformation.* The function $F(\nu)$ expresses the amount of wave at frequency ν contained in $f(x)$. Since we consider $L \to \infty$ in the Fourier series, the frequency interval between adjacent waves tends to zero. It indicates that the sequence of Fourier coefficients $\{a_k\}$ tends to a continuous function $F(\nu)$.

The pair of functions $f(x)$ and $F(\nu)$ is called *Fourier pair,* and it is expressed as $FT[f(x)](\nu) = F(\nu)$ and $FT^{-1}[F(\nu)](x) = f(x)$.

In two dimensional case, Fourier transformation is expressed as follows [1]:

$$F(\nu_x, \nu_y) = \iint_{-\infty}^{\infty} f(x, y) \exp\{-i2\pi(\nu_x x + \nu_y y)\} dxdy. \tag{5}$$

The original (x, y) plane is called real domain, and (ν_x, ν_y) plane yielded by Fourier transformation is called *frequency domain.*

[1] The right side of this equation is divided by N for normalization in some textbooks. This is explained later in the section of the unitary transformation.

Sampling and sampling theorem

Assume an image as a one-dimensional function for simplicity. Let the brightness at position x, i. e. pixel value of pixel at x, be $f(x)$. Extracting the brightness at points arranged regularly at a period is called sampling, as shown in Fig. 1.

Let us suppose a function composed by infinite numbers of Dirac's delta functions arranged at an interval T, as shown in Fig. 2. This is called *comb function,* defined as follows:

$$\mathrm{comb}_T(x) = \sum_{n=-\infty}^{\infty} \delta(x - nT). \tag{6}$$

A sampled digital image from $f(x)$, denoted $f_T(x)$, is expressed as $f(x)$ multiplied by the comb function $\mathrm{comb}_T(x)$, i. e.

$$f_T(x) = f(x)\mathrm{comb}_T(x). \tag{7}$$

Now we consider the Fourier transformation of $f_T(x)$ to find the frequency range of sampled image $f_T(x)$. We apply the following theorem on the Fourier transformation of the product of two functions:

$$FT[f(x)g(x)](\nu) = FT[f(x)](\nu) * FT[g(x)](\nu) \tag{8}$$

where $FT[f(x)]$ denotes the Fourier transform of $f(x)$, and the symbol $*$ denotes convolution, defined as follows:

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(y)g(t-y)dy. \tag{9}$$

The theorem of Eq. (8) states that *the Fourier transform of the product of two functions is equal to the convolution of the Fourier transforms of the two functions* (see Appendix 1 for the proof).

We get from this theorem that the Fourier transformation of Eq. (7) is

$$FT[f_T(x)](\nu) = FT[f(x)](\nu) * FT[\mathrm{comb}_T(x)](\nu). \tag{10}$$

The first term of the right side of Eq. (10) is the Fourier transform of the original function $f(x)$. The second term is the Fourier transform of comb function. We get from a theorem that

$$FT[\mathrm{comb}_T(x)](\nu) = \frac{1}{T}\mathrm{comb}_{1/T}(\nu). \tag{11}$$

(see Appendix 2 for the outline of the proof.) This relationship states that the Fourier transform of a comb function is also a comb function, and the period of the original comb function and that of the transformed comb function in the frequency domain are in invert proportion. Consequently, we get

$$FT[f_T(x)](\nu) = \frac{1}{T}\{FT[f(x)](\nu) * \mathrm{comb}_{1/T}(\nu)\}. \tag{12}$$

What is "convolution with comb function?" We explain it by starting from "convolution with delta function." From Eq. (9), we get

$$f(t) * \delta(t) = \int_{-\infty}^{\infty} f(y)\delta(t-y)dy. \tag{13}$$

At the right side of Eq. (13), y varies from $-\infty$ to ∞. Since $\delta(t-y) = 0$ except $t = y$, the contribution of $\delta(t-y)$ to the integral is zero in this case . Thus we get

$$\begin{aligned} f(t) * \delta(t) &= \int_{-\infty}^{\infty} f(y)\delta(t-y)dy \\ &= \int_{-\infty}^{\infty} f(y)\delta(t-t)dy \\ &= f(t)\int_{-\infty}^{\infty} \delta(0)dy = f(t), \end{aligned} \tag{14}$$

i. e. *the convolution of a function and the delta function is equal to the original function itself.*

Since a comb function is a sequence of the delta functions arrraged at a constant period, the convolution of a function and a comb function is an arrangement of shifted duplications of the original function at a constant period. Consequently, Eq. (12) states that the Fourier transform of $f_T(x)$, which is the sampled version of $f_T(x)$ at period T, is an infinite sequence of shifted duplications of $FT[f_T(x)]$, the Fourier transform of the original $f(x)$, arranged at interval $1/T$. This relationship is illustrated in Fig. 3. Here ν_c is called *cutoff frequency* and indicates the highest frequency contained in the original $f(x)$. As explained in the previous session, if $f(x)$ is a real function, the components of $FT[f(x)]$ lie in the range between $-\nu_c$ and ν_c, since $FT[f(x)](-\nu) \neq 0$ if $FT[f(x)](\nu) \neq 0$.

If the period of comb function in the frequency domain is sufficiently large, as shown in Fig. 4(a), adjacent $FT[f(x)]$'s do not overlap. In this case, the

Fourier transform of the original function, $FT[f(x)]$, can be separated and extracted, i. e. no information of the brightness distribution of the original image is lost by the sampling. However, if the interval of the comb functions in the frequency domain is small, as shown in Fig. 4(b), adjacent $FT[f(x)]$'s overlap. In this case, the original $FT[f(x)]$ cannot be separated and a faulty function will be extracted. This effect is called *aliasing*.

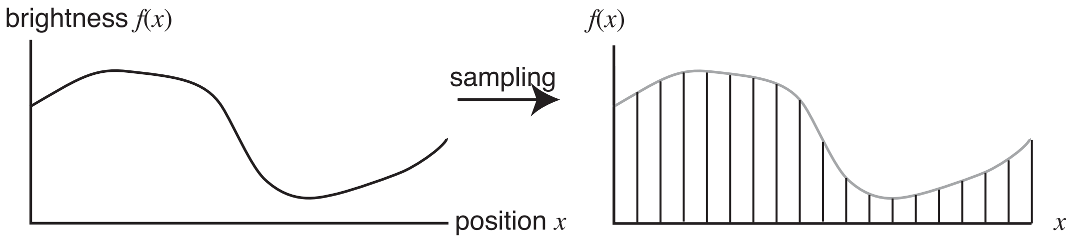

Fig. 1: Sampling.

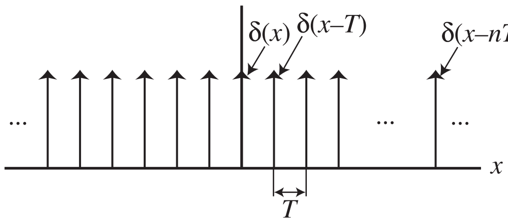

Fig. 2: Comb function.

Since the support of $FT[f(x)]$ is in the range between $-\nu_c$ and ν_c, the period has to be at least $2\nu_c$ for avoiding overlaps of $FT[f(x)]$'s. Since T is a sampling period, $1/T$ denotes the number of samples per unit length, i. e. sampling rate. Consequently, *the original brightness distribution can be reconstructed by a sampled digital image if the sampling rate is more than twice the maximum frequency contained in the original distribution.* This theorem is called *sampling theorem.*

For example, the music compact disc is recorded digitally at sampling rate 44.1kHz, i. e. the signal is sampled 44100 times per second. Thus the maximum frequency that the music CD can correctly reproduce is 22.05kHz. It means that the filtering for cutting off the frequency range higher than 22.05kHz at the recording process is required for avoiding an aliasing.

Appendix 1. Convolution and Fourier transformation

Let F and G be Fourier transforms of real-domain functions f and g, respectively. We get from Eq. (4), the definition of the inverse Fourier transformation, that

$$\begin{aligned} &f(x)g(x) \\ &= \int_{-\infty}^{\infty} F(\nu)\exp(i2\pi\nu x)d\nu \int_{-\infty}^{\infty} G(\mu)\exp(i2\pi\mu x)d\mu \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\nu)G(\nu)d\nu \exp(i2\pi(\nu+\mu)x)d\mu. \end{aligned} \tag{15}$$

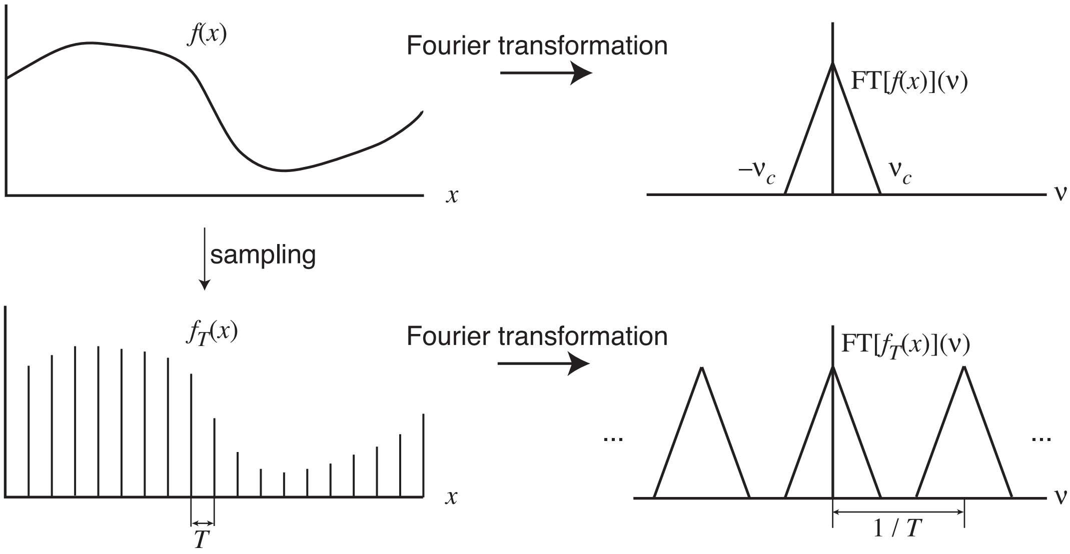

Fig. 3: Sampling and Fourier transformation.

Applying the variable conversion $\nu + \mu = \xi$, we get

$$\begin{aligned} f(x)g(x) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\nu)G(\xi-\nu)d\nu \exp(i2\pi\xi x)d\xi \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [F*G](\xi)\exp(i2\pi\xi x)d\xi \\ &= FT^{-1}[F*G](x) \end{aligned} \tag{16}$$

Equation (8) is obtained by the inverse transformation of Eq. (16).

Appendix 2. Fourier transformation of comb function

From the definition of comb$T(x)$ by Eq. (6), we get that comb$T(x)$ is a periodic function of period T. A periodic function of period T is expressed by a series of sinusoidal functions whose frequency is n/T (n : integer), i. e.

$$\mathrm{comb}_T(x) = \sum_{n=-\infty}^{\infty} a_n \exp(i2\pi\frac{n}{T}x). \tag{17}$$

The coefficient a_n at the frequency n/T is obtained by multiplication of $\exp(-i2\pi\frac{n}{T}x)$ and integration, because of the property of the orthogonal function system. Since it is a periodic function of period T, the range of integration is not $(-\infty, \infty)$ but $[-T/2, T/2]$. Multiplying $1/T$ for normalization, we get

$$\begin{aligned} a_n &= \frac{1}{T}\int_{-T/2}^{T/2} \mathrm{comb}_T(x)\exp(-i2\pi\frac{n}{T}x)dx \\ &= \frac{1}{T}\int_{-T/2}^{T/2} \delta(x)\exp(-i2\pi\frac{n}{T}x)dx \\ &= \frac{1}{T}\exp(-i2\pi\frac{n}{T}\cdot 0) = \frac{1}{T}, \end{aligned} \tag{18}$$

and then we get

$$\mathrm{comb}_T(x) = \frac{1}{T}\sum_{n=-\infty}^{\infty} \exp(i2\pi\frac{n}{T}x). \tag{19}$$

Thus its Fourier transform is as follows:

$$FT[\mathrm{comb}_T(x)](\nu) = \frac{1}{T}\sum_{n=-\infty}^{\infty} FT[\exp(i2\pi\frac{n}{T}x)](\nu). \tag{20}$$

As explained in the previous session, since we get peaks at $\nu = n/T$ in the Fourier transform of $\exp(i2\pi\frac{n}{T}x)$, we get from Eq. (20) that

$$\begin{aligned} FT[\mathrm{comb}_T(x)](\nu) &= \frac{1}{T}\sum_{n=-\infty}^{\infty} \delta(\nu - \frac{n}{T}) \\ &= \frac{1}{T}\mathrm{comb}_{1/T}(\nu). \end{aligned} \tag{21}$$

Reference

H. P. Hsu, Fourier Analysis, Simon & Schuster, 1967（佐藤平八訳，フーリエ解析，森北出版，1979 ISBN4-627-93010-0）.

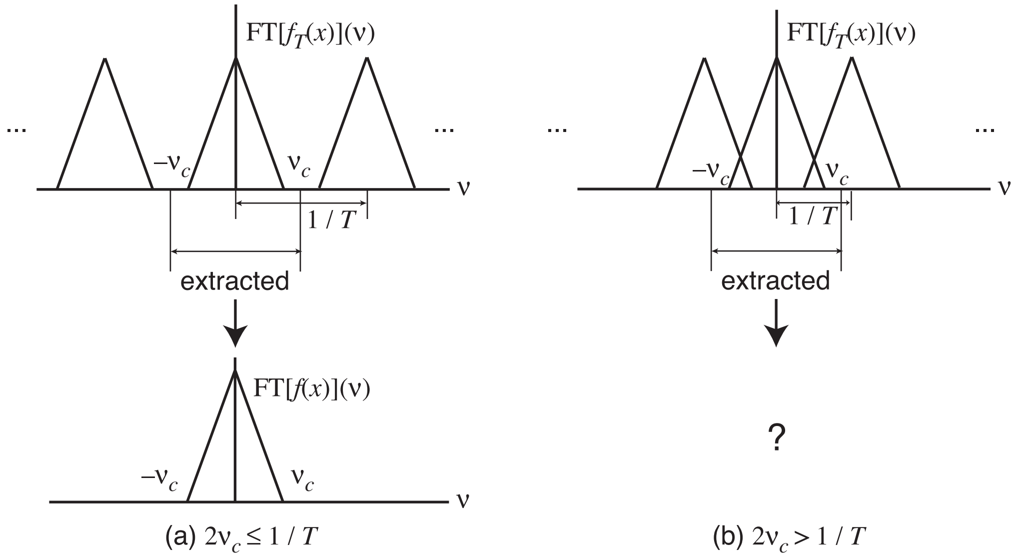

Fig. 4: Sampling theorem.